UDC 004.438

REMARKS ON EXTREMAL AND MAXIMUM DETERMINANT MATRICES WITH MODULI OF REAL ENTRIES ≤ 1

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Purpose: *This note discusses quasi-orthogonal matrices which were first highlighted by J. J. Sylvester and later by V. Belevitch, who showed that three level matrices mapped to lossless telephone connections. The goal of this note is to develop a theory of such matrices based on preliminary research results.* **Methods:** *Extreme solutions (using the determinant) have been established by minimization of the maximum of the absolute values of the elements of the matrices followed by their subsequent classification.* **Results:** *We give the definitions of Balonin–Mironovsky (BM), Balonin–Sergeev (BSM) and Cretan matrices (CM), illustrations for some elementary and some interesting cases, and reveal some new properties of weighing matrices (Balonin– Seberry conjecture). We restrict our attention in this remark to the properties of Cretan matrices depending on their order.* **Practical relevance:** *Web addresses are given for other illustrations and other matrices with similar properties. Algorithms to construct Cretan matrices have been implemented in developing software of the research program-complex.*

Keywords — *Hadamard Matrices, Conference Matrices, Weighing Matrices, Constructions, Balonin–Mironovsky Matrices, Balonin–Sergeev Matrices, Cretan Matrices.*

AMS Subject Classification: *05B20; 20B20.*

Definition 1. A real square matrix $\mathbf{X} = (x_{ij})$ of order n is called *quasi-orthogonal* if it satisfies $\mathbf{X}^T\mathbf{X} = \mathbf{X}\mathbf{X}^T = c\mathbf{I}_n$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix and "T" stands for transposition, c is constant real number. In this and future work we will only use quasi-orthogonal to refer to matrices with real elements, a least one entry in each row and column must be 1. Hadamard matrices are the best known of these matrices with entries from the unit disk [1].

Definition 2. An *Hadamard matrix* of order n is an $n \times n$ matrix with elements 1, −1 such, that $\mathbf{H}^T\mathbf{H} = \mathbf{H}\mathbf{H}^T = n\mathbf{I}_n$, where $\mathbf{I}_n$ is the identity matrix.

The Hadamard inequality [2] says, that Hadamard matrices have maximal determinant for the class of matrices with entries from the unit disk (the moduli of the elements is $|x_{ij}| \leq 1$ by default). Hadamard matrices can only exist for orders 1, 2 and $n = 4t$, t an integer (the so called *Hadamard conjecture*).

The class of *quasi-orthogonal* matrices with maximal determinant and entries from the unit disk may have a very large set of solutions. Different solutions may give the same maximal determinant. Symmetric conference matrices, a particularly important class of 0, ±1 matrices, are the most well known [3].

Definition 3. A *symmetric conference matrix,* $\mathbf{C}$, is an $n \times n$ matrix with elements 0, +1 or −1, satisfying $\mathbf{C}^T\mathbf{C} = \mathbf{C}\mathbf{C}^T = (n - 1)\mathbf{I}_n$.

Conference matrices can only exist if the number $n - 1$ is the sum of two squares. Similar to symmetric conference matrices are quasi-orthogonal matrices $\mathbf{W} = \mathbf{W}(2t, 2t - m)$ of order $n = 2t$, with elements 0, +1 or −1, satisfying $\mathbf{W}^T\mathbf{W} = \mathbf{W}\mathbf{W}^T =$ $= (2t - m)\mathbf{I}_n$. These are called *weighing matrices*. It has been conjectured [4] that for $n = 4t$, there exists a $\mathbf{W} = \mathbf{W}(4t, 4t - m)$ for all integers $0 \leq m \leq 4t$.

Definition 4. The values of the entries of the quasi-orthogonal matrix, $\mathbf{X}$, are called *levels*, so Hadamard matrices are two-level matrices and symmetric conference matrices and weighing matrices are three-level matrices. *Quasi-orthogonal* matrices with maximal determinant of odd orders have been discovered to have a larger number of levels [5].

Definition 5. A *Balonin−Mironovsky* [5] matrix, $\mathbf{A}_n$, of order n, is quasi-orthogonal matrix of maximal determinant. In this remark they are called *BM matrices.*

Conjecture (Balonin, [6, 7]): there are only 5 Balonin−Mironovsky matrices $\mathbf{A}_3$, $\mathbf{A}_5$, $\mathbf{A}_7$, $\mathbf{A}_9$, $\mathbf{A}_{11}$ with $\frac{n+1}{2} \pm m$, $m \leq 1$, levels.

The 2006 paper [5] gave 5 examples of BM matrices. Order 13 was unresolved. During 2006−2011 Balonin and Sergeev carried out many computer experiments to find the absolute maximum of the determinant of $\mathbf{A}_{13}$.

It was speculated [6] that 13 is a critical order for matrices of odd orders with maximal determinant. Starting from this odd order, the number of levels $k >> \frac{n+1}{2}$. An example of a 6-level (by moduli) matrix of even order was found and called Yura's matrix $\mathbf{Y}_{22}$ [8] (Fig. 1, *a*). A student Yura Balonin found this rare solution using DOS−MatLab [8, 9]. The matrix levels are captured by the colour of the squares.

Order $n = 22$ is special, $n - 1$ is not sum of two squares, and a symmetric three level conference matrix does not exist. The two circulant matrix $\mathbf{Y}_{22}$

based on the sequences $\{-f\ b\ a\ -a\ a\ a\ a\ a\ -a\ a\ -a\}$, $\{a\ a\ -a\ -c\ -a\ a\ d\ a\ e\ a\ -a\}$ has elements with moduli $a = 1$, $b = 0.9802$, $c = 0.7845$, $d = 0.6924$, $e = 0.5299$, $f = 0.3076$. It appears similar to a conference matrix of order 22 because of the small value for f. A non optimal determinant version was also found with $f = 0.0055$.

It was then discovered that there is a 22×22 matrix W(22,20) constructed using Golay sequences which gave $\det(W(22,20)) > \det(Y_{22})$ (Fig. 1, *b*).

We note that Golay sequences exist which give $W(2n, 2n - 2)$ with determinant $(2n - 2)^n$ for orders 4, 6, 10, 18, 22, 34, 42, 54, 66, 82, 102, 106, 130, 162, 258, 262, 322. In the cases 22, 34, 66, 106, 130, 162, 210, 322 there is no corresponding conference matrix [3].

Conjecture I (Balonin–Seberry, 2014): Suppose a W($2n$, $2n - 1$) does not exist. Suppose a W($2n$, $2n - 2$) exists. Then the quasi-orthogonal matrix with maximal determinant is constructed using the W($2n$, $2n - 2$).

Conjecture II (Balonin–Seberry, 2014): Suppose a W($2n$, $2n - 1$) does not exist. Suppose that W($2n$, k) is the weighing matrix with largest k that exists, then W($2n$, k) will give a quasi-orthogonal matrix with near maximal determinant.

For order 58 Balonin found [10] a quasi-orthogonal matrix Y_{58} with only a few levels and determinant $2 \cdot 10^{50}$, the weighing matrices W(58, k), $k = 54, 55, 56, 57$, do not exist. The weighing matrix W(58,53) has determinant 10^{50}, so conjecture I only applies for W($2n$, $2n - 2$) matrices (Fig. 2, *a*, *b*).
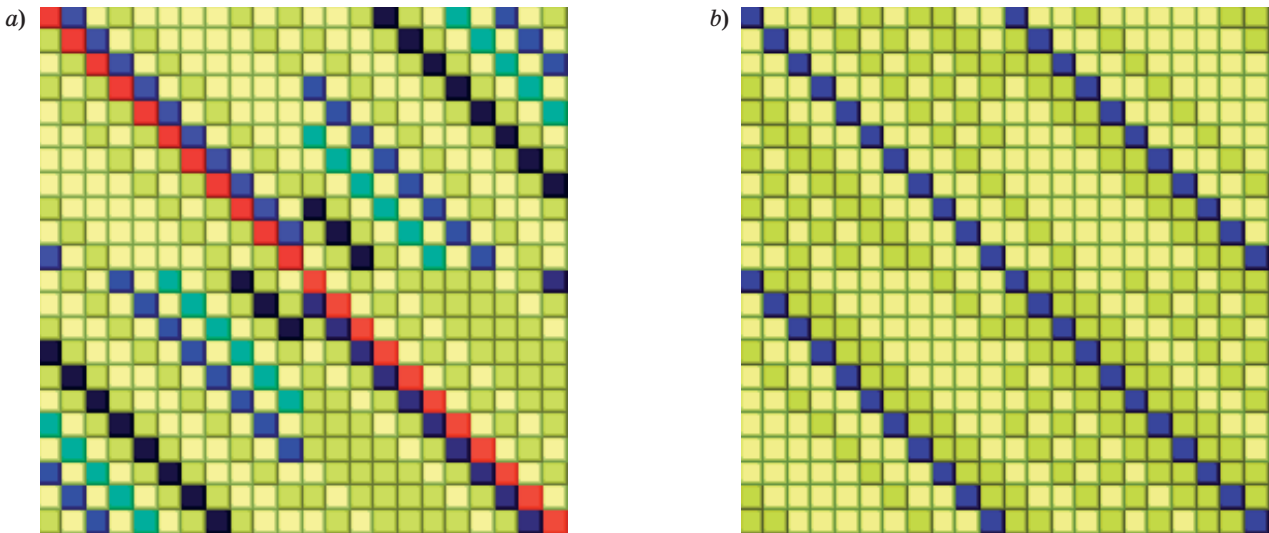

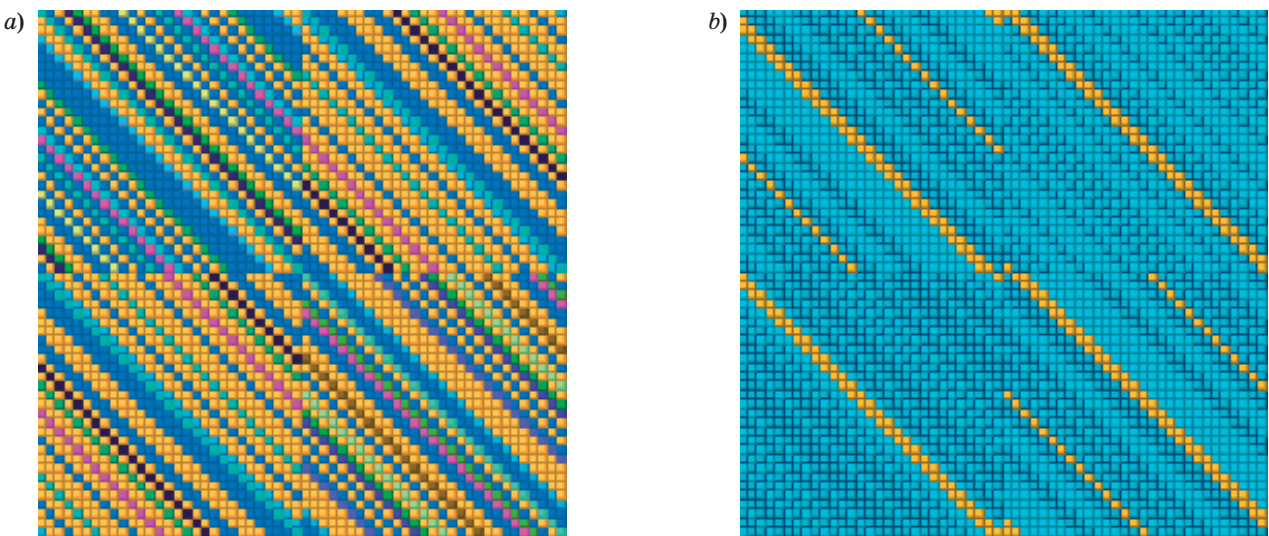**Fig. 1.** Yura's matrix Y_{22} (*a*) and a weighing matrix W(22,20) (*b*)

Fig. 2. A low number of levels matrix of order 58 (*a*) and a weighing matrix W(58,53) (*b*)

The absence of a solution with a low number of levels for $n \geq 13$, led Balonin and Sergeev to search for and classify quasi-orthogonal matrices with other properties [5, 6, 11–13].

Definition 6. A quasi-orthogonal matrix with extremal or fixed properties: global or local extremum of the determinant, saddle points, the minimum number of levels, or matrices with fixed numbers of levels is called a *Balonin–Sergeev* matrix. They are called here BSM-matrices.

A Balonin–Mironovsky matrix is a Balonin–Sergeev matrix with the absolute maximum determinant. Balonin–Sergeev matrices with fixed numbers of levels were first mentioned during a conference in Crete, so we will call them *Cretan matrices* (*CM-matrix*).

Definition 7. A *Cretan matrix*, **X**, of order n, which has indeterminate entries, x_1, x_2, x_3, x_4, …, x_k is said to have k levels.

It satisfies $\mathbf{X}^T\mathbf{X} = \mathbf{X}\mathbf{X}^T = \omega(n)\mathbf{I}_n$, $\mathbf{I}_n$ the identity matrix, $\omega(n)$ the weight, that give a number of equations, called the *CM-equations*, which make **X** quasi-orthogonal when the variables (indeterminates) are replaced by real elements with moduli $|x_{ij}| \leq 1$.

The $\mathbf{X}^T\mathbf{X} = \mathbf{X}\mathbf{X}^T$ have diagonal entries the weight $\omega(n)$ and off diagonal entries 0. *CM-matrices* can be defined by a function $\omega(n)$ or functions $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$, …, $x_k(n)$. We write $CM(n; k; \omega(n);$ determinant) as shorthand.

Notation: When the variable (indeterminate) entries, x_1, x_2, x_3, x_4, …, x_k occur s_1, s_2, s_3, s_4, …, s_k times in each row and column, we write $CM(n; k; s_1, s_2, s_3, s_4, …, s_k; \omega(n);$ determinant) as shorthand.

A review and questions of existence are discussed in [7, 13, 14].

Balonin and Sergeev concluded [7, 13] that the resolution of the question of the existence of quasi-orthogonal matrices and their generalizations discussed here depends on the order [15]:

— for $n = 4t$, t an integer, at least 2 levels, a, $-b$, $|a| = |b|$, are needed;

— for $n = 4t - 1$, at least 2 levels, $a = 1$, $-b$, $b < a$, are needed;

— for $n = 4t - 2$, at least 2 levels, $a = 1$, $-b$, $b < a$, are needed for a two block circulant construction;

— for $n = 4t - 3$, at least 3 levels, $a = 1$, $-b$, c, $b < a, c < a$, are needed.

Definitions and examples of different types of *Cretan matrices* will be discussed in future papers.

Acknowledgements

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