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TWO-CIRCULANT GOLDEN RATIO MATRICES

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Purpose: *This paper considers two-level quasi-orthogonal matrices, complementing the Mersenne and Euler matrices belonging to the class of Hadamard type matrices, which were first highlighted by J. Hadamard and V. Belevitch. The goal of this note is to develop a theory of such matrices based on preliminary research results. The definitions are provided.* **Methods:** *Extreme solutions (using the determinant) have been established by minimization of the absolute values of the elements of the matrices followed their subsequent classification.* **Results:** *We give the definitions of a section and a layer of quasi-orthogonal matrices. An example of continuous matrices with varying levels is used to show that the branch of golden ratio matrices is closely associated with the Hadamard and Belevitch matrices. Commentary on the applied aspects of the two-circulant golden ratio matrices and illustrations for some elementary and some interesting cases of Fermat, Mersenne and Euler matrices are provided.* **Practical relevance:** *Web addresses are given for other illustrations and other matrices with similar properties. Algorithms to construct golden ratio matrices have been implemented in developing software of the research program-complex.*

Keywords — *Quasi-Orthogonal Matrices, Hadamard Matrices, Belevitch Matrices, Mersenne Matrices, Euler Matrices, Golden Section, Golden Ratio Matrices.*
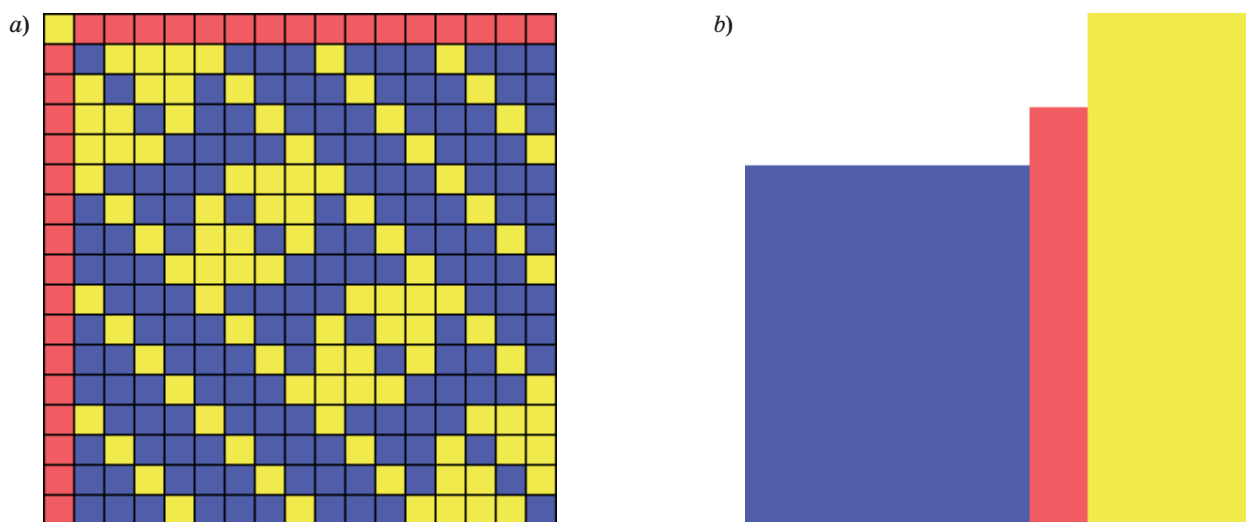
Introduction

An analysis of *Mersenne* matrix existence [1, 2] has raised the question of quasi-orthogonal matrices belonging to the Hadamard type matrix family [3], the special cases of which are *Belevitch* (C-matrices) [4, 5], *Hadamard* [6, 7], *Mersenne* [8, 9], *Euler* [10], and *Fermat* matrices [11]. The matrices are listed in descending order for $n = 4k - d$, where $d = 0, 1, 2, 3, k > 0$ is integer.

A *quasi-orthogonal matrix*, of order n, is a square matrix $\mathbf{A}$, with $|a_{ij}| \leq 1$ in each column (and row), with maximum modulus 1, has $\mathbf{A}^T\mathbf{A} = \omega(n)\mathbf{I}$, with $\mathbf{I}$ the identity matrix and $\omega(n)$ is the *weight*.
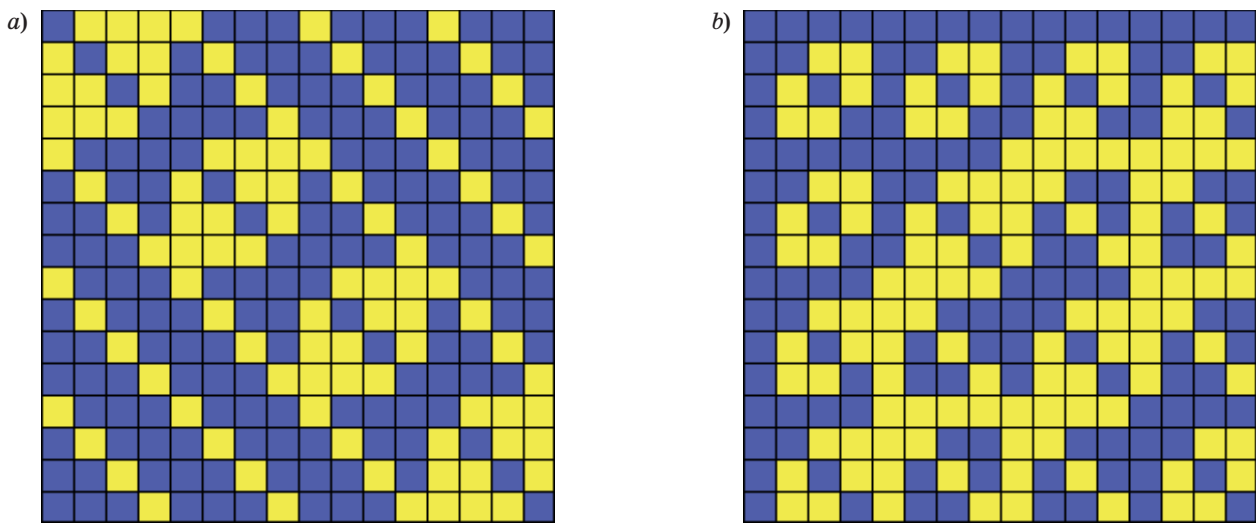
The values of the entries we will call "levels" of the matrix. An Hadamard matrix with entries $\{1, -1\}$ is a two-level matrix. A Mersenne matrix with entries $\{1, -b\}$, $0 < b < 1$ is also a two-level matrix. Now matrices, themselves, can belong to a layer.

Definition 1. In this paper a *matrix layer* is a set of quasi-orthogonal matrices with known functions for entries describing their dependence on $n = 4k - d$ for some d and all possible $k > 0$.
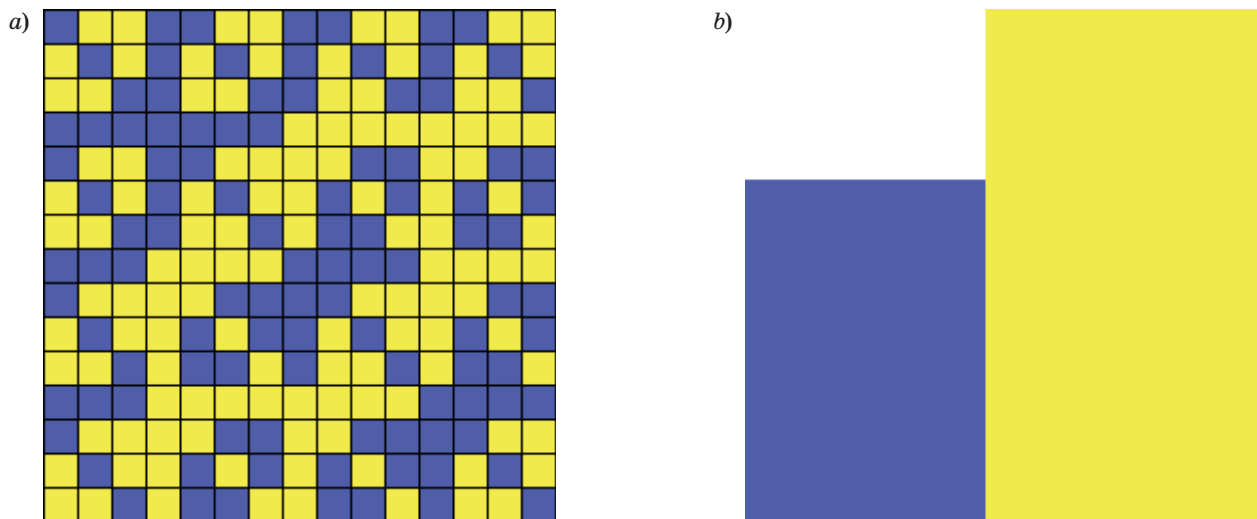
A Mersenne matrix, of order n, has negative entries $-b$, described by some function $b = f(n)$ and determined for all orders $n = 4k - 1$. Any fixed (non-varying) Mersenne matrix belongs to this layer. In the same way, Hadamard and Euler matrices with sizes $n = 4k - d$, $d = 0, 2$, as described in [1–3],
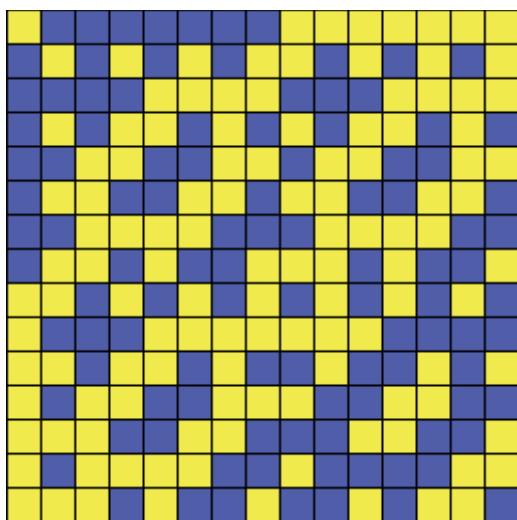

■ *Fig. 1.* Fermat matrix $\mathbf{F}_{17}$ (*a*) and histogram of moduli of its elements (*b*)

■ *Fig. 2*. Hadamard matrix $\mathbf{H}_{16}$ before (*a*) and after (*b*) normalization

■ *Fig. 3*. Mersenne matrix $\mathbf{M}_{15}$ (*a*) and histogram of moduli of its elements (*b*)
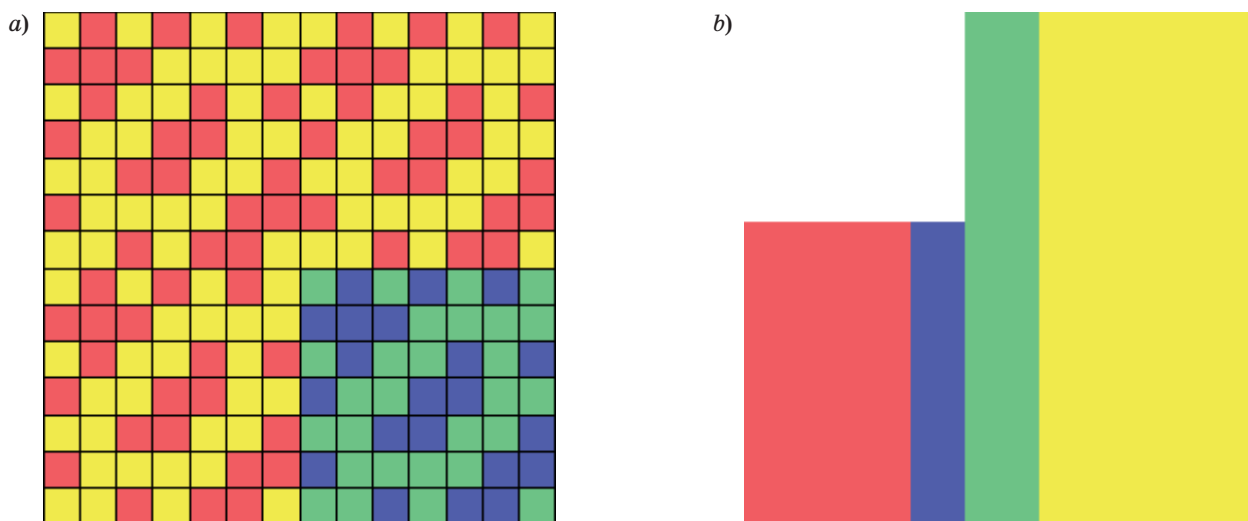

■ *Fig. 4*. Normalized Mersenne $\mathbf{M}_{15}$

belong to layers. Fermat matrices [11] do not form such layer, as their level functions are defined within a narrow set of values $n = 2^k + 1$ for even and some odd values of integer k.

Definition 2. In this paper a *section* is a set of quasi-orthogonal matrices of different layers, which depend on $n = 4k - d$ for some k and all possible $d = 0, 1, 2, 3$.

A particular (wider) section can be expanded by Fermat matrices using the same principle — the Fermat matrix (Fig. 1) with size $4k + 1$ can be used to find the corresponding Hadamard matrix (Fig. 2) with size $4k$ (the main order of the section). The Hadamard matrix can then be used to find a Mersenne matrix (Fig. 3, 4) with size $4k - 1$. This last matrix can be used to find an Euler (Fig. 5) matrix with size $4k - 2$ [3].

■ *Fig. 5.* Euler matrix $\mathbf{E}_{14}$ (*a*) and histogram of moduli of its elements (*b*)

Matrices with Few Levels

The matrices mentioned above are the manifestation of a mathematical object, described by its layers and sections [3]. The existence of any matrix in a section entails the existence of all other matrices of the same section because these matrices are mutually dependent.

Besides Hadamard matrices with entries $\{1, -1\}$ and similar to them Mersenne matrices with entries $\{1, -b\}$, $0 < b < 1$, there are other matrices with small numbers of levels. The Euler matrix $\mathbf{E}_n$ [3, 11] (shown in Fig. 5) is a square matrix of order $n = 4k - 2$ with entries $\{1, -1, b, -b\}$, where $\mathbf{E}_n^{\mathbf{T}}\mathbf{E}_n = \xi\,\mathbf{I}_n$, $\mathbf{I}_n$ is an identity matrix, $\xi = \dfrac{(n+2)+(n-2)b^2}{2}$, and $b = \dfrac{1}{2}$, when $n = 6$, in other cases $b = \dfrac{q - \sqrt{8q}}{q - 8}$, where $q = n + 2$.

The number of levels is an important characteristic of a matrix set. The low number of levels does not guarantee existence of Belevitch matrices (conference matrices) [4, 5], they do not exist for order $n = 4k - 2$, if $n - 1$ is not sum of two squares.

The number of matrix levels increases with the value d in the interval $n = 4k - d$. Hadamard matrices have single level (by modulus of elements) matrices as the elements are 1 or −1. Mersenne matrices are two-level matrices; Euler matrices are four-level matrices. All these matrices have some minimal number of levels guaranteeing their existence for pre chosen orders [3].

Many sets of quasi-orthogonal matrices with low numbers of levels do not belong to a layer, they are special orthogonal per columns (Hadamard type) matrices: conference matrices with three levels of entries $\{0, 1, -1\}$ are defined for orders shared with the bigger family of four levels Euler matrices. Paley [7] noted that any Hadamard matrix (or quasi-orthogonal matrix respectively) can be used to give matrix of the double size using the Sylvester algorithm. These we call these Sylvester constructions.

In this case, a new matrix branch appears: it does not intersect with any of the previous branches. Paley's observation induces us to study *artifact matrices* from the orthogonal matrix family (the Hadamard family), including the two-circulant *golden ratio matrix*. This is considered in this paper. The golden ratio matrix [13] and the two-circulant *golden ratio matrix* of order 10 lead to G-matrices of orders $n = 10 \cdot 2^k$, these sizes 10, 20, 40, 80, 160, 320, 640, etc. hold a special place in image processing algorithms.

Continuous Matrices

Continuous matrices are different from previously observed section matrices of the orthogonal (Hadamard) family, their level functions depend on more than one argument n. Therefore, for each n they generate not one, but a continuum of quasi-orthogonal matrices, described by a parametric dependence. This possibility follows from the interpretation of orthogonal or quasi-orthogonal matrix as a table of vector projections of the required orthogonal basis. We use optimal to denote matrices with maximal determinant. This allows us to get non-varying matrices for this continuum, known as orthogonal (Hadamard) matrices [6, 7].

Sub-optimal solutions are known for M-matrices [14, 15] with a small number of levels. Fig. 6 shows a continuous M-matrix $\mathbf{M}_{10}$.

$$\mathbf{M}_{10} = \begin{pmatrix} g & a & -c & -c & a & -a & a & b & b & a \\ a & g & a & -c & -c & a & -a & a & b & b \\ -c & a & g & a & -c & b & a & -a & a & b \\ -c & -c & a & g & a & b & b & a & -a & a \\ a & -c & -c & a & g & a & b & b & a & -a \\ -a & a & b & b & a & -g & -a & c & c & -a \\ a & -a & a & b & b & -a & -g & -a & c & c \\ b & a & -a & a & b & c & -a & -g & -a & c \\ b & b & a & -a & a & c & c & -a & -g & -a \\ a & b & b & a & -a & -a & c & c & -a & -g \end{pmatrix}.$$
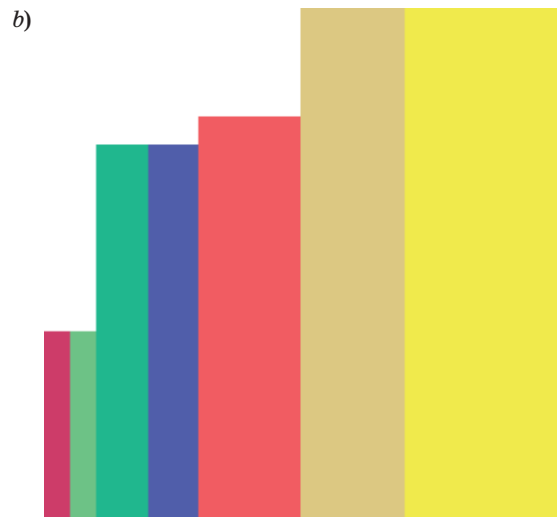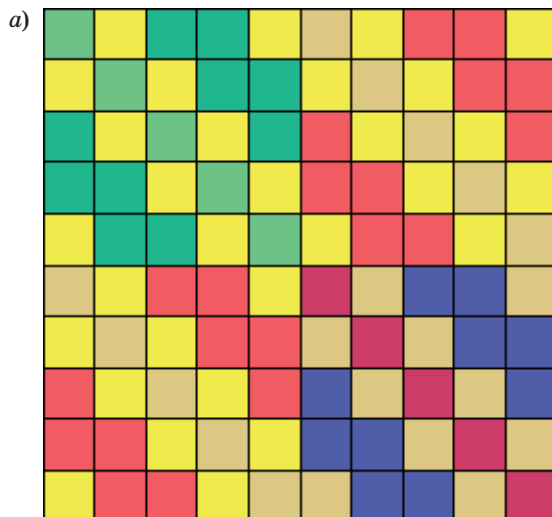
The upper module level from set of levels $a \geq b \geq c \geq g$ is $a = 1$. The second and the third levels depend on the lower level g as $b^2 + 2(b - 1) + 2(g - c) + c^2 = 0$, $c = 1/(g + 1)$.

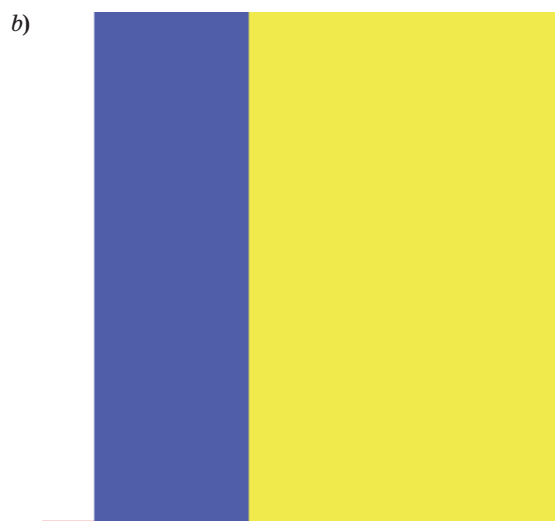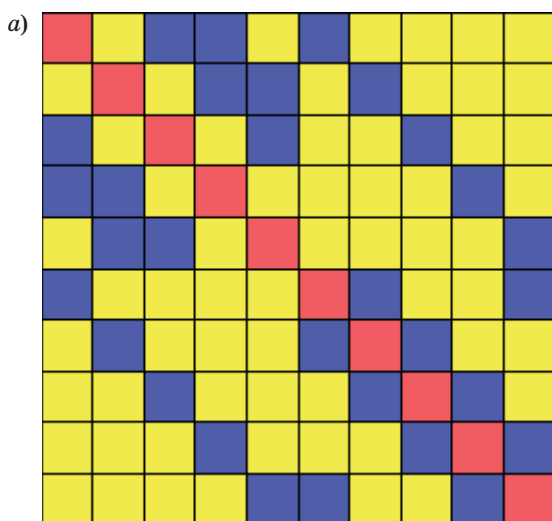The coloured matrix portrait represents the structure and levels of entries — every level has its own colour.

The continuous matrix $\mathbf{M}_{10}$ is a matrix with a low number of changeable levels and is notable by its solutions: two bounds (Fig. 7, 8) of a continuum.

One solution (see Fig. 7) is the two-circulant Belevitch matrix $\mathbf{C}_{10}$ since when $b = c = a = 1$ we have $g = 0$.
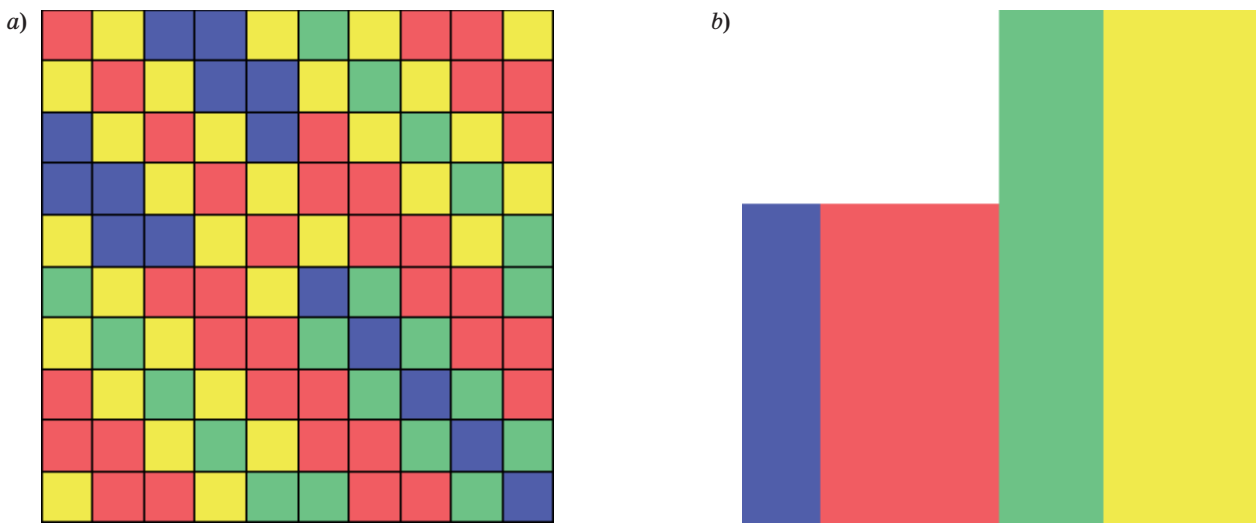
We call the second solution (see Fig. 8), with $b = c = g < a = 1$, as two-circulant *golden ratio matrix* $\mathbf{G}_{10}$.
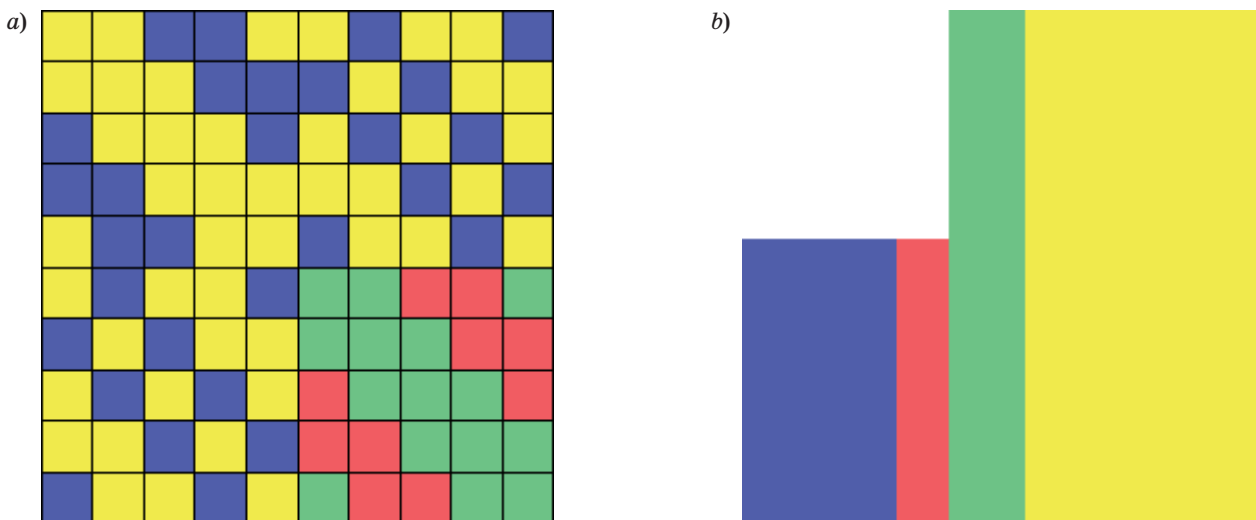

■ *Fig. 6.* Portrait of matrix $\mathbf{M}_{10}$ (*a*) and histogram of moduli of its elements (*b*)

■ *Fig. 7.* Belevitch matrix $\mathbf{C}_{10}$ (*a*) and histogram of moduli of its elements (*b*)

■ *Fig. 8.* Golden ratio matrix $\mathbf{G}_{10}$ (*a*) and histogram of moduli of its elements (*b*)

■ *Fig. 9.* Euler matrix $\mathbf{E}_{10}$ (*a*) and histogram of modules of its elements (*b*)

$$\mathbf{G}_{10} = \begin{pmatrix} g & a & -g & -g & a & -a & a & g & g & a \\ a & g & a & -g & -g & a & -a & a & g & g \\ -g & a & g & a & -g & g & a & -a & a & g \\ -g & -g & a & g & a & g & g & a & -a & a \\ a & -g & -g & a & g & a & g & g & a & -a \\ -a & a & g & g & a & -g & -a & g & g & -a \\ a & -a & a & g & g & -a & -g & -a & g & g \\ g & a & -a & a & g & g & -a & -g & -a & g \\ g & g & a & -a & a & g & g & -a & -g & -a \\ a & g & g & a & -a & -a & g & g & -a & -g \end{pmatrix}.$$

It is distinguished by the equation $g^2 + g - 1 = 0$, well known by its irrational roots called the *golden ratio* in the Fibonacci number theory. In this case we are interested in the lower level $g = 0.618...$

that proportional to the inversion of 1.618... . Matrices with such elements were for the first time provided in **[13]**.
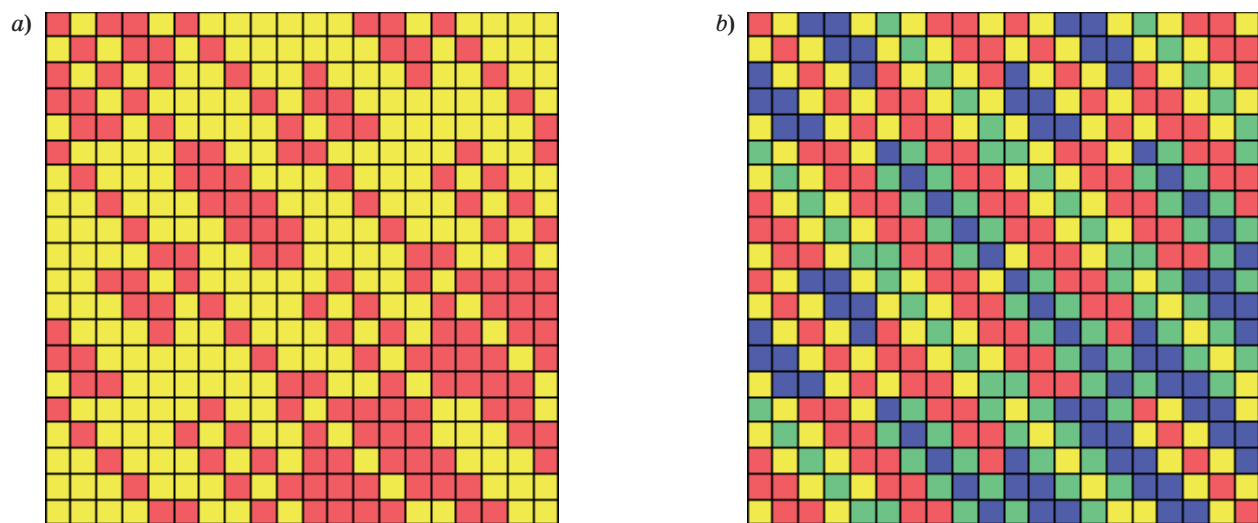
Let's note that histograms for golden ratio matrix $\mathbf{G}_{10}$ and Euler matrix $\mathbf{E}_{10}$ are similar to each other (see Fig. 8, 9), they both are two-level (by modulus) matrices.

Golden ratio matrices

Consequently, all golden ratio matrices are defined on orders $n = 10 \cdot 2^k$.

For them, as for all Hadamard family matrices, matrix $\mathbf{G}_{10}$ is the starting point for the sequence of matrices, found by iterations

$$\mathbf{G}_{2n} = \begin{pmatrix} \mathbf{G}_n & \mathbf{G}_n \\ \mathbf{G}_n & -\mathbf{G}_n \end{pmatrix}.$$

a) *b)*

■ *Fig. 10*. Hadamard matrix $\mathbf{H}_{20}$ (*a*) and golden ratio matrix $\mathbf{G}_{20}$ (*b*)

The value of modulus level g is constant. This implies, that golden ratio matrices and Hadamard-type matrices are two boundary solutions of a continuum matrix (Fig. 10).

Acknowledgements

Conclusion

This paper describes a golden ratio matrix $\mathbf{G}_{10}$ and sequence of such G-matrices, represented by the example $\mathbf{G}_{20}$. These matrices are closely associated with Belevitch and Hadamard-type matrices, their specific structures and algorithms to find them.

Golden ratio matrices, represented by $\mathbf{G}_{10}$, connect with Belevitch matrices as bounds of continuum. The latter (coexisting with Euler matrices) have no solutions for the orders 22, 34, 58 and so on. So golden ratio matrices do not have a layer by the determination.

The range of application of mathematical models as orthogonal bases is wide [3]. There is a curious idea to use the continuous matrix as a model of phase transformations taking place during the crystallization of cooled alloys [16].

The special boundary points of the continuous matrix can explain the patterns, observed in the tests. The two level golden ratio matrix can be a model reflecting the details of crystal structure [17]. The peculiarities of the quasi-crystal problem are present here — the dichotomy of elements, associated with the golden ratio level and specific orders [18]. Matrix models may be calculated and used to predict the existence of new materials [19].

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