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Articles

Theoretical foundations of digital vector Fourier analysis of two-dimensional signals padded with zero samples

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Introduction: *The practice of using Fourier-processing of finite two-dimensional signals (including images), having confirmed its effectiveness, revealed a number of negative effects inherent in it. A well-known method of dealing with negative effects of Fourier-processing is padding signals with zeros. However, the use of this operation leads to the need to provide information control systems with additional memory and perform unproductive calculations.* ***Purpose:*** *To develop new discrete Fourier transforms for efficient and effective processing of two-dimensional signals padded with zero samples.* ***Method:*** *We have proposed a new method for splitting a rectangular discrete Fourier transform matrix into square matrices. The method is based on the application of the modulus comparability relation to order the rows (columns) of the Fourier matrix.* ***Results:*** *New discrete Fourier transforms with variable parameters were developed, being a generalization of the classical discrete Fourier transform. The article investigates the properties of Fourier transform bases with variable parameters. In respect to these transforms, the validity has been proved for the theorems of linearity, shift, correlation and Parseval's equality. In the digital spectral Fourier analysis, the concepts of a parametric shift of a two-dimensional signal, and a parametric periodicity of a two-dimensional signal have been introduced. We have estimated the reduction of the required memory size and the number of calculations when applying the proposed transforms, and compared them with the discrete Fourier transform.* ***Practical relevance:*** *The developed discrete Fourier transforms with variable parameters can significantly reduce the cost of Fourier processing of two-dimensional signals (including images) padded with zeros.*

Keywords — *discrete Fourier transform, two-dimensional signal, Fourier processing, effects of discrete Fourier transform, basis, variable parameter.*

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Introduction

Fourier-processing of finite discrete two-dimensional (FDTD) signals (including images) in informational control (IC) systems is the most important method for studying processes and analyzing information [1–8]. The theoretical basis of Fourier-processing of FDTD signals is two-dimensional direct and inverse discrete Fourier transforms (2D DFT, 2D IDFT) [9–15] which can be represented in form of:

— algebraic form 2D DFT

$$S_{N_1,N_2}(k_1, k_2) = \frac{1}{N_1 N_2}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1, n_2) W_{N_1}^{k_1 n_1}\cdot W_{N_2}^{k_2 n_2},$$

where $S_{N_1,N_2}(k_1, k_2)$ are coefficients (bins) 2D DFT; $k_1 = \overline{0, (N_1-1)}$, $k_2 = \overline{0, (N_2-1)}$ are spatial frequencies; $x(n_1, n_2)$ is 2D signal; $n_1 = \overline{0, N_1-1}$, $n_2 = \overline{0, N_2-1}$; $W_{N_1}^{k_1 n_1} = \exp\left(-j\frac{2\pi}{N_1}k_1 n_1\right)$; $W_{N_2}^{k_2 n_2} = \exp\left(-j\frac{2\pi}{N_2}k_2 n_2\right)$;

— algebraic form 2D IDFT

$$x(n_1, n_2) = \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} S_{N_1,N_2}(k_1, k_2) W_{N_1}^{-k_1 n_1}\cdot W_{N_2}^{-k_2 n_2}.$$

The practice of using DFT and 2D DFT, on the one hand, confirmed their efficiency, on the other hand, revealed a number of effects: aliasing effect, scalloping effect, picket fence effect, negatively affecting on the results of analysis and information processing [16–21].

To eliminate these negative effects of DFT and 2D DFT, the zero-padding operation (ZPO) the FDTD signal has found wide application. ZPO can significantly reduce the impact of negative effects on the results of Fourier-processing [22, 23]. However, effective use of ZPO requires solving the problem of Fourier-processing of FDTD of this kind of signals. The essence of the problem lies in the fact that in Fourier-processing of signals subjected to ZPO, on the one hand, it is necessary to provide the corresponding IC systems with a significant amount of additional memory, on the other hand, the IC systems must perform unproductive computations with zero samples, which significantly increases time of Fourier-processing. The paper proposes and investigates new discrete Fourier transforms, which allow efficient and effective analysis and processing of two-dimensional signals padded with zero samples.

The role of the zero-padding operation of FDTD signals in two-dimensional Fourier-processing

The systems analysis of Fourier-processing theory of FDTD signals made it possible to formulate its axiomatic basic provisions:

— determination of FDTD signals on a finite two-dimensional reference plane, which is interpreted as a two-dimensional fundamental period $SA_{N_1 \times N_2}$ (2D period). 2D period is set by horizontal and vertical periods;

— determination of the shift of a two-dimensional discrete signal in the form of a cyclic shift, carried out by cyclic permutation of its samples on the final reference area $SA_{N_1 \times N_2}$;

— definition of a complete two-dimensional basis system

$$\mathrm{def}_{N_1,N_2}(k_1,\, n_1,\, k_2,\, n_2) = W_{N_1}^{k_1 n_1} \cdot W_{N_2}^{k_2 n_2},$$

where $n_1 = \overline{0, N_1 - 1}$; $n_2 = \overline{0, N_2 - 1}$; $k_1 = \overline{0, (N_1 - 1)}$; $k_2 = \overline{0, (N_2 - 1)}$.

As a result of the discreteness and periodicity of 2D signals in the spatial domain, the periodicity and discreteness of 2D Fourier spectra in the spatial-frequency domain, the mathematical operations of convolution and correlation are cyclical. However, the analysis, design, and modeling of isoplanatic systems requires the results of linear operations with 2D signals.

The method, which allows obtaining the results of linear operations using cyclic operations, consists in expanding the reference regions with zero samplex of the convoluted signals by applying ZPO to them.

If the reference area $SA_{V_1 \times V_2}$ of signal $x(n_1, n_2)$ and the reference area $SA_{Q_1 \times Q_2}$ of signal $y(n_1, n_2)$ are specified, then the size of the reference area, padded with zeros to obtain linear convolution $h_{linear}(n_1, n_2)$, should be

$$SA_{(V_1+Q_1)\times(V_2+Q_2)},$$

where $n_1 = \overline{0,\, (V_1 + Q_1 - 1)}$; $n_2 = \overline{0,\, (V_2 + Q_2 - 1)}$.

And the size of the reference area for obtaining linear correlation $C_L(n_1, n_2)$ should be

$$SA_{2V_1 \times 2V_2},$$

where $n_1 = \overline{0,\, (2V_1 - 1)}$; $n_2 = \overline{0,\, (2V_2 - 1)}$.

Therefore, the algorithm for obtaining 2D linear convolution based on 2D cyclic convolution consists of the following operations:

1. Pad 2D signals $x(n_1, n_2)$ and $y(n_1, n_2)$ with Q_1, Q_2 and V_1, V_2 zero samples respectively, which sets new 2D signals $x_0(n_1, n_2)$, $y_0(n_1, n_2)$ with horizontal N_2 and vertical N_1 periods according to the ratios

$$N_1 \ge (V_1 + Q_1 - 1);\; N_2 \ge (V_2 + Q_2 - 1).$$

2. Perform 2D DFT of 2D signals $x_0(n_1, n_2)$ and $y_0(n_1, n_2)$:

$$x_0(n_1,\, n_2) \xrightarrow{F} X_{0,N_1,N_2}(k_1, k_2);$$

$$y_0(n_1,\, n_2) \xrightarrow{F} Y_{0,N_1,N_2}(k_1, k_2),$$

where $\xrightarrow{F}$ is the 2D DFT execution symbol.

3. Perform 2D IDFT product

$$X_{0,N_1,N_2}(k_1, k_2) \cdot Y_{0,N_1,N_2}(k_1, k_2).$$

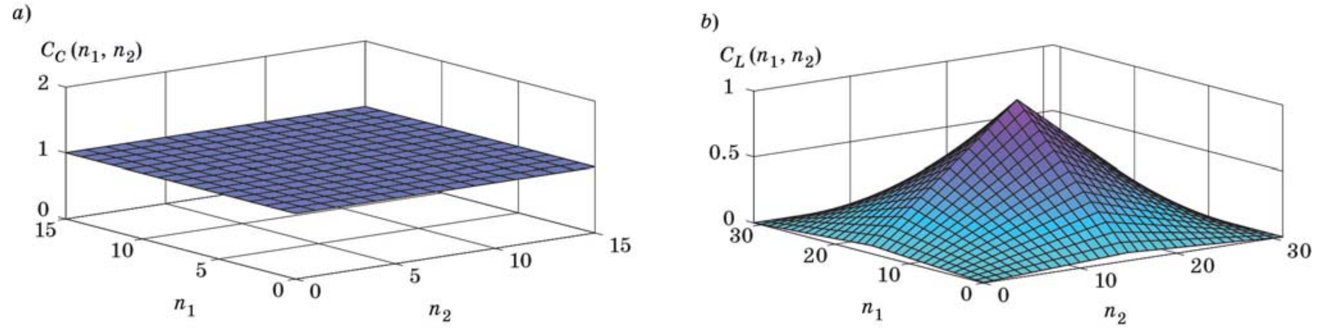
The algorithm for obtaining a linear 2D correlation function based on a cyclic 2D correlation function is easy to obtain from the previous algorithm. Fig. 1, *a* and *b* illustrates the differences between cyclic $C_C(n_1, n_2)$ and linear $C_L(n_1, n_2)$ correlation functions of a finite unit 2D signal.

According to the two-dimensional version of the Wiener — Khinchin theorem, Fourier transform of the linear 2D correlation function allows one to obtain the energy spectrum of a 2D signal. There is a so-called direct method for obtaining the energy spectrum of a 2D signal $x(n_1, n_2)$, bypassing the stage of obtaining the correlation function:

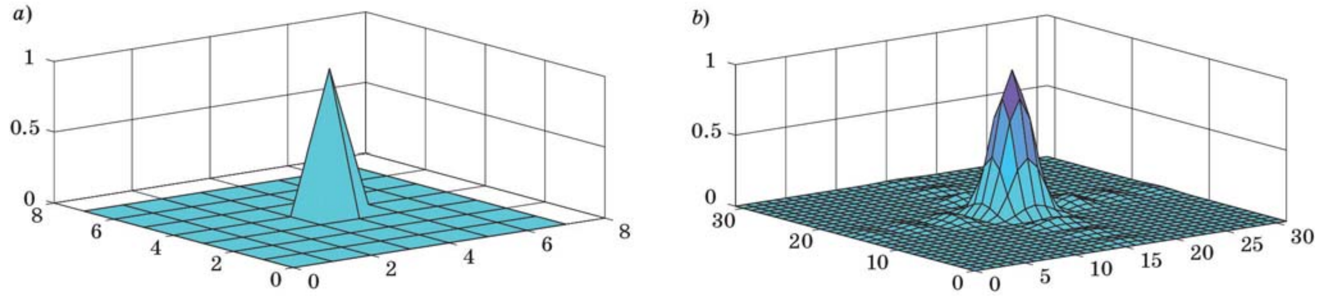
$$G_{N_1,N_2}(k_1,\, k_2) = N_1 N_2 \left| S_{N_1,N_2}(k_1,\, k_2) \right|^2.$$

A significant drawback of this definition of the energy spectrum of a 2D signal $x(n_1, n_2)$ is insufficient detailing $G_{N_1,N_2}(k_1,\, k_2)$, for example, to fulfill the conditions of Pugachev canonical signal decomposition. The method of increasing the detail $G_{N_1,N_2}(k_1,\, k_2)$ is carried out by padding the 2D signal $x(n_1, n_2)$ with zeros at least twice. Fig. 2, *a* and *b* illustrates the detailing of the energy spectrum of a finite single 2D signal.

As noted in the introduction, the effective application of the ZPO requires a solution to the problem of Fourier-processing of FDTD signals padded with zero samples. The essence of the problem lies in the fact that in Fourier-processing of signals subjected to the ZPO, on the one hand, it is necessary to provide the corresponding IC system with a significant additional amount of RAM (storage), on the other hand, IC system must perform a lot of non-productive calculations with zero samples, which significantly increases the time of Fourier-processing.



■ ***Fig. 1.*** Cyclic (*a*) and linear (*b*) correlation 2D functions of a finite single 2D signal



■ ***Fig. 2.*** Energy spectrum of a finite single 2D signal with zero frequency in the center of the spectrum: *a* — a finite single 2D signal $N_1 = 8$, $N_2 = 8$; *b* — a finite single 2D signal, padded with zero samples to $N_1 = 32$, $N_2 = 32$

Let us consider a generalization of 2D DFT in the form of a 2D DFT with a variable parameter, which makes it possible to efficiently analyze and process two-dimensional signals subjected to ZPO.

Two-dimensional DFT with variable parameter

Let two 2D signals be given: a signal $\mathbf{X}_{N_1\times N_2}$ and a signal $\mathbf{O}_{N_1\times N_2}$ with zero samples.

To perform the linear transformations considered in the previous section, it is necessary to pad (supplement) the horizontal period of the 2D signal $\mathbf{X}_{N_1\times N_2}$ with $(r_2 - 1)$ zero matrices $\mathbf{O}_{N_1\times N_2}$, which leads to a block matrix:

$$\begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & . & . & (r_2-1) \end{array} \\ \mathbf{X}_{N_1\times(N_2 r_2)} = & \left[\begin{array}{ccccc} \mathbf{X}_{N_1\times N_2} & \mathbf{O}_{N_1\times N_2} & . & . & \mathbf{O}_{N_1\times N_2} \end{array}\right]. \end{array} \tag{1}$$

Taking into account the separability property of the 2D DFT, Fourier transform of a signal $\mathbf{X}_{N_1\times(N_2 r_2)}$ in matrix form can be represented as

$$\mathbf{S}^{k_1,k_2}_{N_1\times(N_2 r_2)} = \frac{1}{N_1 N_2}\mathbf{F}^{(2)}_{N_1\times N_1}\left[\mathbf{X}_{N_1\times(N_2 r_2)}\mathbf{F}^{(1)}_{(N_2 r_2)\times(N_2 r_2)}\right], \tag{2}$$

where

$$\mathbf{F}^{(2)}_{N_1\times N_1} = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & . & . & (N_1-1) \end{array} \quad n_1 \\ \begin{array}{c} 0 \\ 1 \\ . \\ . \\ (N_1-1) \\ k_1 \end{array} & \left[\begin{array}{ccccc} W_{N_1}^{0\cdot 0} & W_{N_1}^{0\cdot 1} & . & . & W_{N_1}^{0\cdot(N_1-1)} \\ W_{N_1}^{1\cdot 0} & W_{N_1}^{1\cdot 1} & . & . & W_{N_1}^{1\cdot(N_1-1)} \\ . & . & . & . & . \\ . & . & . & . & . \\ W_{N_1}^{(N_1-1)\cdot 0} & W_{N_1}^{(N_1-1)\cdot 1} & . & . & W_{N_1}^{(N_1-1)(N_1-1)} \end{array}\right] \end{array};$$

$$\mathbf{X}_{N_1\times(N_2r_2)}=\begin{array}{cc} & \begin{array}{ccccccc} 0 & .. & (N_2-1) & N_2 & .. & (N_2r_2-1) & n_2 \end{array}\\ \begin{array}{c} 0\\ 1\\ .\\ .\\ (N_1-2)\\ (N_1-1)\\ n_1 \end{array} & \left[\begin{array}{cccccc} x(0,\ 0) & .. & x(0,\ (N_2-1)) & 0 & .. & 0\\ x(1,\ 0) & .. & x(1,\ (N_2-1)) & 0 & .. & 0\\ . & .. & . & . & .. & .\\ . & .. & . & . & .. & .\\ x((N_1-2),\ 0) & .. & x((N_1-2),\ (N_2-1)) & 0 & .. & 0\\ x((N_1-1),\ 0) & .. & x((N_1-1),\ (N_2-1)) & 0 & .. & 0 \end{array}\right] \end{array};$$

$$\mathbf{F}^{(1)}_{(N_2r_2)\times(N_2r_2)}=\begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & .\ . & (N_2r_2-1) & k_2 \end{array}\\ \begin{array}{c} 0\\ 1\\ .\\ .\\ (N_2r_2-1)\\ n_2 \end{array} & \left[\begin{array}{cccc} W_{N_2}^{0\cdot 0} & W_{N_2}^{0\cdot 1} & .\ . & W_{N_2}^{0\cdot(N_2r_2-1)}\\ W_{N_2}^{1\cdot 0} & W_{N_2}^{1\cdot 1} & .\ . & W_{N_2}^{1\cdot(N_2r_2-1)}\\ . & . & .\ . & .\\ . & . & .\ . & .\\ W_{N_2}^{(N_2r_2-1)\cdot 0} & W_{N_2}^{(N_2r_2-1)\cdot 1} & .\ . & W_{N_2}^{(N_2r_2-1)(N_2r_2-1)} \end{array}\right] \end{array}. \quad (3)$$

Let us interrogate the structure of matrix equation (2). It is easy to see that the multiplication of matrices $\mathbf{X}_{N_1\times(N_2r_2)}$ and $\mathbf{F}^{(1)}_{(N_2r_2)\times(N_2r_2)}$ leads to a rectangular matrix $\mathbf{Y}_{N_1\times(N_2r_2)}$. A matrix $\mathbf{Y}_{N_1\times(N_2r_2)}$ can be interpreted as the product of a matrix $\mathbf{X}_{N_1\times N_2}$ by a matrix $\mathbf{F}^{(1)}_{N_2\times(N_2r_2)}$:

$$\mathbf{Y}_{N_1\times(N_2r_2)}=\mathbf{X}_{N_1\times N_2}\cdot\mathbf{F}^{(1)}_{N_2\times(N_2r_2)},$$

where

$$\mathbf{X}_{N_1\times N_2}=\begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & ... & (N_2-1) & n_2 \end{array}\\ \begin{array}{c} 0\\ 1\\ .\\ .\\ (N_1-2)\\ (N_1-1)\\ n_1 \end{array} & \left[\begin{array}{cccc} x(0,\ 0) & x(0,\ 1) & ... & x(0,\ (N_2-1))\\ x(1,\ 0) & x(1,\ 1) & ... & x(1,\ (N_2-1))\\ . & . & ... & .\\ . & . & ... & .\\ x((N_1-2),\ 0) & x((N_1-2),\ 1) & ... & x((N_1-2),\ (N_2-1))\\ x((N_1-1),\ 0) & x((N_1-1),\ 1) & ... & x((N_1-1),\ (N_2-1)) \end{array}\right] \end{array}; \quad (4)$$

$$\mathbf{F}^{(1)}_{N_2\times(N_2r_2)}=\begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & .\ . & (N_2r_2-1) & k_2 \end{array}\\ \begin{array}{c} 0\\ 1\\ .\\ .\\ (N_2-1)\\ n_2 \end{array} & \left[\begin{array}{cccc} W_{N_2}^{0\cdot 0} & W_{N_2}^{0\cdot 1} & .\ . & W_{N_2}^{0\cdot(N_2r_2-1)}\\ W_{N_2}^{1\cdot 0} & W_{N_2}^{1\cdot 1} & .\ . & W_{N_2}^{1\cdot(N_2r_2-1)}\\ . & . & .\ . & .\\ . & . & .\ . & .\\ W_{N_2}^{(N_2-1)\cdot 0} & W_{N_2}^{(N_2-1)\cdot 1} & .\ . & W_{N_2}^{(N_2-1)(N_2r_2-1)} \end{array}\right] \end{array}. \quad (5)$$

Comparing matrices $\mathbf{F}^{(1)}_{(N_2r_2)\times(N_2r_2)}$ (3) and $\mathbf{F}^{(1)}_{N_2\times(N_2r_2)}$ (5), we may see that the matrix $\mathbf{F}^{(1)}_{N_2\times(N_2r_2)}$ is the result of truncating $N_2(r_2-1)$ the rows of the matrix $\mathbf{F}^{(1)}_{(N_2r_2)\times(N_2r_2)}$. According to [22], “we denote the set of matrix column numbers by A...:

$$A: A=\{0,\ 1,\ 2,\ \ldots,\ (N_2r_1-1)\}.$$

We apply to the set A the relation of comparability modulo r_2.

It is known that the relation of comparability in modulus m is an equivalence relation and has the properties of reflexivity, symmetry and transitivity."

Also we know that "the relation of comparability modulo r_2 divides the set A into r_2 classes of residues modulo r_2:

$$A_0 = \{0,\ r_2,\ \ldots,\ (N_2-1)r_2\};$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_{(r_2-1)} = \{(r_2-1),\ \ldots,\ (N_2 r_2 - 1)\};$$
$$A_i \neq \varnothing;\ A_i \bigcap_{i\neq j} A_j = \varnothing;\ \bigcup_{i=0}^{r_1-1} A_i = A. \qquad (6)$$

The matrix $\mathbf{F}^{(1)}_{N_2\times(N_2 r_2)}$, applying the partition (6) of the set A into r_2 residue classes modulo r_2, can be represented in the form of r_2 matrices of size $N_2 \times N_2$" [22]:

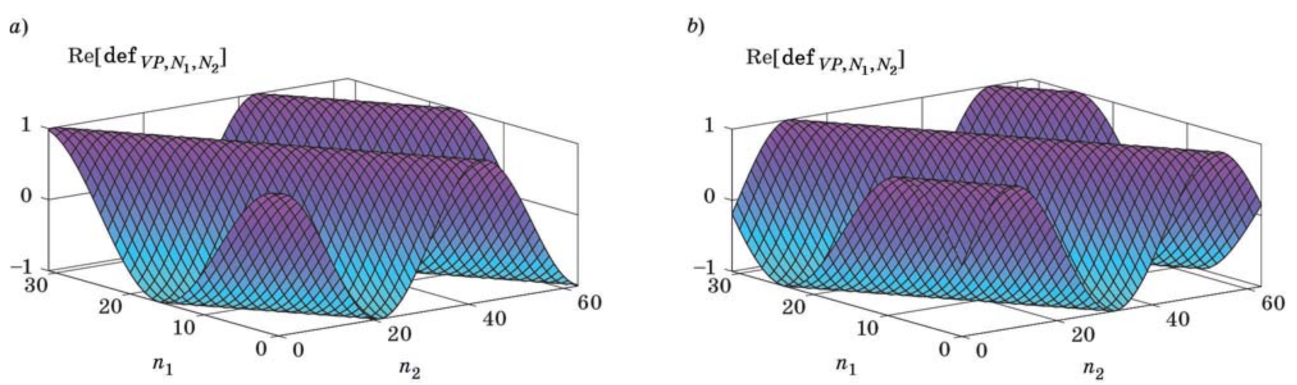
$$\mathbf{F}^{(1)}_{N_2\times N_2,\theta_2} = \begin{array}{c|cccc|c} & 0 & 1 & .\ . & (N_2-1) & k_2 \\ 0 & W_{N_2}^{0\cdot(0+\theta_2)} & W_{N_2}^{0\cdot(1+\theta_2)} & .\ . & W_{N_2}^{0\cdot(N_2-1+\theta_2)} & \\ 1 & W_{N_2}^{1\cdot(0+\theta_2)} & W_{N_2}^{1\cdot(1+\theta_2)} & .\ . & W_{N_2}^{1\cdot(N_2-1+\theta_2)} & \\ . & . & . & .\ . & . & \\ . & . & . & .\ . & . & \\ (N_2-1) & W_{N_2}^{(N_2-1)(0+\theta_2)} & W_{N_2}^{(N_2-1)(1+\theta_2)} & .\ . & W_{N_2}^{(N_2-1)(N_2-1+\theta_2)} & \\ n_2 & & & & & \end{array}, \qquad (7)$$

where $\theta_2 = 0;\ 1/r_2,\ \ldots,\ (r_2-1)/r_2$.

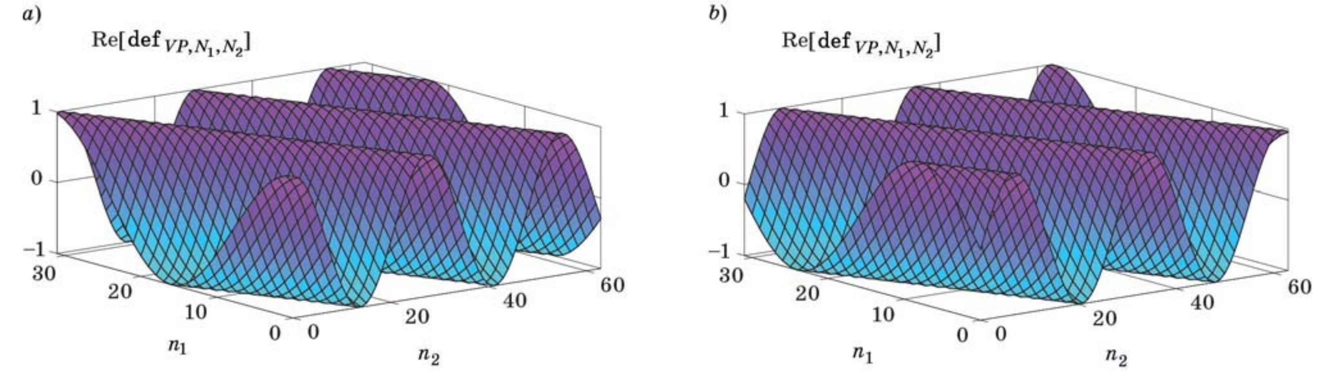
Discrete two-dimensional exponential functions of the form

$$\mathrm{def}_{HP,N_1,N_2}(k_1,\ n_1,\ k_2,\ n_2,\ \theta_2) = W_{N_1}^{k_1 n_1}\cdot W_{N_2}^{(k_2+\theta_2)n_2} = \left[\exp\left(-j\frac{2\pi}{N_1}k_1 n_1\right)\right]\cdot\left[\exp\left(-j\frac{2\pi}{N_2}(k_2+\theta_2)n_2\right)\right] =$$
$$= \left[\cos\left(\frac{2\pi}{N_1}k_1 n_1\right) - j\sin\left(\frac{2\pi}{N_1}k_1 n_1\right)\right]\cdot\left[\cos\left(\frac{2\pi}{N_2}(k_2+\theta_2)n_2\right) - j\sin\left(\frac{2\pi}{N_2}(k_2+\theta_2)\right)n_2\right] =$$
$$= \cos\left(\frac{2\pi}{N_1}k_1 n_1 + \frac{2\pi}{N_2}(k_2+\theta_2)n_2\right) - j\sin\left(\frac{2\pi}{N_1}k_1 n_1 + \frac{2\pi}{N_2}(k_2+\theta_2)n_2\right), \qquad (8)$$

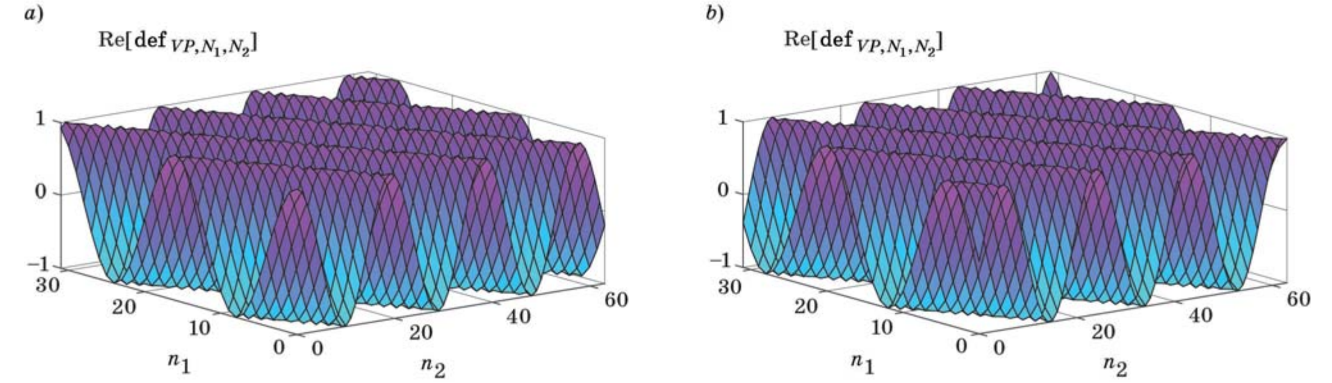
where $k_1 = \overline{0,\ N_1-1}$; $k_2 = \overline{0,\ N_2-1}$; $0 \le \theta_2 < 1$, will be called two-dimensional discrete exponential functions with a variable parameter — 2D DEF-VP (Figs. 3–5).



■ ***Fig. 3.*** Two-dimensional exponential function with variable parameter at $N_1 = 32$, $N_2 = 64$; $k_1 = 1$, $k_2 = 1$; $\theta_2 = 1/2$: a — a real part; b — an imaginary part



■ ***Fig. 4.*** Two-dimensional exponential function with variable parameter at $N_1 = 32$, $N_2 = 64$; $k_1 = 1$, $k_2 = 2$; $\theta_2 = 1/3$: *a* — a real part; *b* — an imaginary part



■ ***Fig. 5.*** Two-dimensional exponential function with variable parameter at $N_1 = 32$, $N_2 = 64$; $k_1 = 2$, $k_2 = 3$; $\theta_2 = 1/3$: *a* — a real part; *b* — an imaginary part

The introduction of discrete exponential functions with a variable parameter makes it possible to generalize the concept of periodicity of the DEF-VP system. Recall that the periodicity of the DEF system in the classical DFT is understood as a periodic continuation of the DEF system outside the interval of N samples. Moreover, the system of discrete basis functions in the classical DFT does not contain discontinuities. In the case of discrete Fourier transform with a variable parameter (DFT-VP) (9), for the DEF-VP system to be inseparable, the periodicity should be understood as parametric periodicity. The parametric periodicity of discrete exponential functions with a variable parameter is understood as their periodic continuation with rotation in complex space by an angle of $2\pi\theta$. Note that the introduced concept of parametric periodicity is valid for 1D and 2D real and complex functions.

Consider the main properties of two-dimensional discrete exponential functions of 2D DEF-VP.

Main properties of 2D DEF-VP

Each of the two-dimensional discrete exponential functions with a variable parameter has its own spatial frequencies k_1, k_2, which determine its place in a particular basic system. The set of 2D DEF-VP makes its basic system of two-dimensional discrete Fourier transform with a variable parameter (2D DFT-VP) in space $\mathbf{I}_2^N$.

For each value of the parameter θ_2 we can say that:

1. 2D DEF-VP are complex functions by definition.

2. The basis system 2D DEF-VP is a generalization of the basis system 2D DEF and is equal to it at $\theta_2 = 0$.

3. 2D DEF-VP are two-dimensional functions of four equivalent variables k_1, k_2, n_1, n_2, and one variable parameter θ_2:

$$\mathrm{def}_{HP,N_1,N_2}(k_1,\ n_1,\ k_2,\ n_2,\ \theta_2) = W_{N_1}^{k_1 n_1} \cdot W_{N_2}^{(k_2+\theta_2)n_2}.$$

4. 2D DEF-VP are periodic in variables k_1 and n_1 with a period N_1 and a variable with a period N_2:

$$\mathrm{def}_{HP,N_1,N_2}((k_1 \pm lN_1),\ (n_1 + qN_1),\ (k_2 \pm mN_2),\ n_2,\ \theta_2) = \mathrm{def}_{HP,N_1,N_2}(k_1,\ n_1,\ k_2,\ n_2,\ \theta_2),$$

where l, m, q are integers.

5. 2D DEF-VP are parametrically periodic in a variable n_2 with a period N_2:

$$\mathrm{def}_{HP,N_1,N_2}(k_1,\, n_1,\, k_2,\, n_2 \pm pN_2,\, \theta_2) = \mathrm{def}_{VP,N_1,N_2}(k_1,\, n_1,\, k_2,\, n_2,\, \theta_2)\cdot W_{N_1}^{\theta_2 N_2 p},$$

where p is integer.

A parametric shift of a two-dimensional signal $X_{N_1\times N_2}$ in the horizontal direction is understood as a two-dimensional cyclic parametric shift of the form of

$$\mathbf{C}_{H.Sh} = \begin{array}{c} 0 \\ 1 \\ 2 \\ . \\ . \\ (N_2-1) \end{array} \begin{bmatrix} [\mathbf{X}_{N_1\times N_2}]^{\mathrm{T}}\cdot \mathbf{H}^{0}_{H.Sh,N_2\times N_2} \\ [\mathbf{X}_{N_1\times N_2}]^{\mathrm{T}}\cdot \mathbf{H}^{1}_{H.Sh,N_2\times N_2} \\ [\mathbf{X}_{N_1\times N_2}]^{\mathrm{T}}\cdot \mathbf{H}^{2}_{H.Sh,N_2\times N_2} \\ . \qquad . \\ . \qquad . \\ [\mathbf{X}_{N_1\times N_2}]^{\mathrm{T}}\cdot \mathbf{H}^{(N_2-1)}_{H.Sh,N_2\times N_2} \end{bmatrix},$$

where $\mathbf{H}^{0}_{H.Sh,N_2\times N_2}$ is two-dimensional identity matrix, expression $\mathbf{H}^{m}_{H.Sh,N_2\times N_2}$, $m=\overline{0,\,(N_2-1)}$ means raising to the power m of the matrix of two-dimensional parametric shift:

$$\mathbf{H}_{H.Sh,N_2\times N_2,\theta_2} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ . \\ . \\ . \\ (N_2-2) \\ (N_2-1) \\ n_2 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & . & . & . & (N_2-2) & (N_2-1) & n_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & . & . & . & & 0 \\ 0 & 0 & 0 & . & . & . & & 0 \\ 0 & 0 & 0 & . & . & . & & 0 \\ . & . & . & . & . & . & & . \quad . \\ . & . & . & . & . & . & & . \quad . \\ . & . & . & . & . & . & & . \quad . \\ 0 & 0 & 0 & . & . & . & 0 & 1 \\ \exp(-2\pi\theta_2) & 0 & 0 & . & . & . & 0 & 0 \end{bmatrix} \\ \\ \end{array}.$$

6. The basis system 2D DEF-VP in the variables k_1, k_2 is not multiplicative:

$$\mathrm{def}_{HP,N_1,N_2}(k_1,\, n_1,\, k_2,\, n_2,\, \theta_2)\cdot \mathrm{def}_{HP,N_1,N_2}(k_3,\, n_1,\, k_4,\, n_2,\, \theta_2) \neq$$
$$\neq \mathrm{def}_{HP,N_1,N_2}((k_1+k_3)_{\mathrm{mod}\,N_1},\, n_1,\, (k_2+k_4)_{\mathrm{mod}\,N_2},\, n_2,\, \theta_2).$$

7. The basis system 2D DEF-VP in the variables n_1, n_2 is multiplicative:

$$\mathrm{def}_{HP,N_1,N_2}(k_1,\, n_1,\, k_2,\, n_2,\, \theta_2)\cdot \mathrm{def}_{H,N_1,N_2}(k_1,\, n_3,\, k_2,\, n_4,\, \theta_2) =$$
$$= \mathrm{def}_{HP,N_1,N_2}(k_1,\, (n_1+n_3)_{\mathrm{mod}\,N_1},\, k_2,\, (n_2+n_4)_{\mathrm{mod}\,N_2},\, \theta_2).$$

8. Average value of 2D DEF-IP with spatial frequencies $k_1 \neq 0$, $k_2 \neq 0$ is equal to zero:

$$\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} \mathrm{def}_{HP,N_1,N_2}(k_1,\, n_1,\, k_2,\, n_2,\, \theta_2) = \sum_{n_1=0}^{N_1-1} W_{N_1\times N_2}^{N_2 k_1 n_1}\left[\sum_{n_2=0}^{N_2-1} W_{N_1\times N_2}^{N_2(k_2+\theta_1)n_2}\right] =$$
$$= \left[\frac{1-W_{N_1}^{k_1 N_1}}{1-W_{N_1}^{k_1}}\right]\left[\frac{1-W_{N_2}^{(k_2+\theta_2)N_2}}{1-W_{N_2}^{k_2}}\right] = 0.$$

9. The basic system 2D DEF-VP is an orthogonal basis system with respect to variables k_1, k_2:

$$\frac{1}{N_1 N_2}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} W_{N_1\times N_2}^{(N_2 k_1 n_1 + N_1(k_2+\theta_2)n_2)} \times$$
$$\times W_{N_1\times N_2}^{(N_2 k_3 n_1 + N_1(k_4+\theta_2)n_2)*} = \begin{cases} 1, \text{ if } k_1 = k_3,\ k_2 = k_4 \\ 0, \text{ if } k_1 \neq k_3,\ k_2 \neq k_4 \end{cases};$$

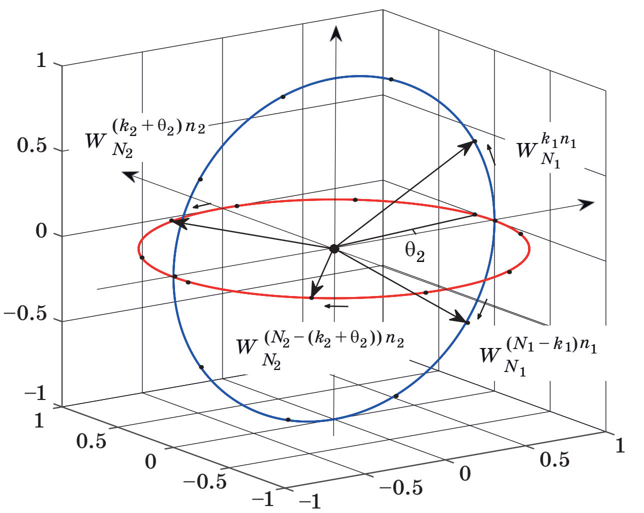
and with respect to variables n_1, n_2:

$$\frac{1}{N_1 N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} W_{N_1\times N_2}^{(N_2 k_1 n_1 + N_1(k_2+\theta_2)n_2)} \times$$
$$\times W_{N_1\times N_2}^{(N_2 k_1 n_3 + N_1(k_2+\theta_2)n_4)*} = \begin{cases} 1, \text{ if } n_1 = n_3,\ n_2 = n_4 \\ 0, \text{ if } n_1 \neq n_3,\ n_2 \neq n_4 \end{cases},$$

where the symbol * means complex conjugation.

10. 2D DEF-VP can be represented by two unit vectors, which represent $W_{N_1}^{k_1 n_1}$ and $W_{N_2}^{(k_2+\theta_2)n_2}$. The unit vectors rotate discontinuously (discretely) in perpendicular complex planes. On the interval N_1, the unit vector which displays $W_{N_1}^{k_1 n_1}$, passes the angle of $2\pi k_1$ radians, making k_1 revolutions, and on the interval N_2, the unit vector, representing $W_{N_2}^{(k_2+\theta_2)n_2}$, passes the angle $2\pi(k_2+\theta_2)$ radians, making $(k_2+\theta_2)$ revolutions. The unit vectors representing the complex conjugate DEF-VP:

$$W_{N_1}^{-k_1 n_1} = W_{N_1}^{(N_1-k_1)n_1} \text{ and}$$
$$W_{N_2}^{-(k_2+\theta_2)n_2} = W_{N_2}^{(N_2-(k_2+\theta_2))n_2}$$



■ ***Fig. 6.*** Representation of a two-dimensional discrete exponential function with a variable parameter in the spatial domain

and make $(N_1 - k_1)$ and $(N_2 - (k_2+\theta_2))$ revolutions respectively.

Figure 6 illustrates such a 2D DEF-VP representation, where the angles $2\pi k_1/N_1$ and $2\pi(k_2+\theta_2)/N_2$ are marked with the corresponding points.

11. The basis system 2D DEF-VP is complete in space $\mathbf{I}_2^N$.

Expansion in basis systems of the form (8) is defined as a 2D DFT-VP. Algebraic form of 2D DFT-VP

$$S_{N_1,N_2}(k_1, k_2, \theta_2) =$$
$$= \frac{1}{N_1 N_2}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x(n_1, n_2) W_{N_1}^{k_1 n_1}\cdot W_{N_2}^{(k_2+\theta_2)n_2}, \quad (9)$$

where k_1, k_2 are spatial frequencies, $k_1 = \overline{0, (N_1-1)}$, $k_2 = \overline{0, (N_2-1)}$; θ_2 is a parameter, $0 \le \theta_2 < 1$; $x(n_1, n_2)$ — two-dimensional signal, $n_1 = \overline{0, N_1-1}$, $n_2 = \overline{0, N_2-1}$; $S_{N_1,N_2}(k_1, k_2, \theta_2)$ are bins of 2D DFT-VP (two-dimensional vector spatial-frequency spectrum of the signal $x(n_1, n_2)$ in the basic 2D DEF-VP system).

The algebraic form of direct 2D DFT-VP, taking into account the property of separability of the kernel (core) of 2D DFT-VP, can be represented as

$$S_{N_1,N_2}(k_1, k_2, \theta_2) =$$
$$= \frac{1}{N_1}\sum_{n_1=0}^{N_1-1} W_{N_1}^{k_1 n_1}\left[\frac{1}{N_2}\sum_{n_2=0}^{N_2-1} x(n_1, n_2) W_{N_2}^{(k_2+\theta_2)n_2}\right]. \quad (10)$$

It can be seen that formula (10) makes it possible to step-by-step calculation of the direct 2D DFT-VP by the method of sequential calculation of two DFT-P (parametric DFT). Note that the calculation of the DFT-P can be carried out by methods of parametric fast Fourier transform (FFT-P) [1].

There is an inverse 2D DFT-VP (2D IDFT-VP):

$$x(n_1, n_2) =$$
$$= \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} S_{N_1,N_2}(k_1, k_2, \theta_2) W_{N_1}^{-k_1 n_1}\cdot W_{N_2}^{-(k_2+\theta_2)n_2},$$

where $n_1 = \overline{0, N_1-1}$; $n_2 = \overline{0, N_2-1}$.

Using the separability property of the 2D DFT-VP kernel, we can introduce the matrix form of the direct 2D DFT-VP:

$$\mathbf{S}_{N_1\times N_2,\theta_2} = \frac{1}{N_1}\mathbf{F}_{N_1\times N_1}^{(2)}\cdot\frac{1}{N_2}\left[\mathbf{X}_{N_1\times N_2}\cdot\mathbf{F}_{N_2\times N_2,\theta_2}^{(1)}\right],$$

where $0 \le \theta_2 < 1$;

$$
\mathbf{F}^{(2)}_{N_1\times N_1} =
\begin{array}{c}
\\ 0 \\ 1 \\ . \\ . \\ (N_1-1) \\ k_1
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & . & . & (N_1-1) \end{array} \quad n_1 \\
\left[\begin{array}{ccccc}
W_{N_1}^{0\cdot 0} & W_{N_1}^{0\cdot 1} & . & . & W_{N_1}^{0\cdot(N_1-1)} \\
W_{N_1}^{1\cdot 0} & W_{N_1}^{0\cdot 1} & . & . & W_{N_1}^{1\cdot(N_1-1)} \\
. & . & . & . & . \\
. & . & . & . & . \\
W_{N_1}^{(N_1-1)\cdot 0} & W_{N_1}^{(N_1-1)\cdot 1} & . & . & W_{N_1}^{(N_1-1)(N_1-1)}
\end{array}\right] \\
\end{array};
$$

$\mathbf{X}_{N_1\times N_2}$ is given by relation (4);

$$
\mathbf{F}^{(1)}_{N_2\times N_2,\theta_2} =
\begin{array}{c}
\\ 0 \\ 1 \\ . \\ . \\ (N_2-1) \\ n_2
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & . & . & (N_2-1) \end{array} \quad k_2 \\
\left[\begin{array}{ccccc}
W_{N_2}^{0\cdot(0+\theta_2)} & W_{N_2}^{0\cdot(1+\theta_2)} & . & . & W_{N_2}^{0\cdot(N_2-1+\theta_2)} \\
W_{N_2}^{1\cdot(0+\theta_2)} & W_{N_2}^{1\cdot(1+\theta_2)} & . & . & W_{N_2}^{1\cdot(N_2-1+\theta_2)} \\
. & . & . & . & . \\
. & . & . & . & . \\
W_{N_2}^{(N_2-1)(0+\theta_2)} & W_{N_2}^{(N_2-1)(1+\theta_2)} & . & . & W_{N_2}^{(N_2-1)(N_2-1+\theta_2)}
\end{array}\right] \\
\end{array}. \qquad (11)
$$

We note the difference between matrices (11) and (7), which lies in the nature of the parameter θ_2 change.

The inverse 2D DFT-VP in matrix form is determined by the matrix equation

$$\mathbf{X}_{N_1\times N_2} = \frac{1}{N_1}\mathbf{F}^{(2)*}_{N_1\times N_1}\cdot\frac{1}{N_2}[\mathbf{S}_{N_1\times N_2,\theta_2}\cdot\mathbf{F}^{(1)*}_{N_2\times N_2,\theta_2}],$$

where $0 \le \theta_2 < 1$.

It can be shown that the theorems of linearity, shift, correlation and Parseval's equality are valid for 2D DFT-VP. For 2D DFT-VP, similar to 2D DFT, the concepts of power spectrum $P_{N_1,N_2}(k_1, k_2, \theta_2)$ and energy spectrum $G_{N_1,N_2}(k_1, k_2, \theta_2)$ can be introduced

$$P_{N_1,N_2}(k_1, k_2, \theta_2) = \left|S_{N_1,N_2}(k_1, k_2, \theta_2)\right|^2; \; G_{N_1,N_2}(k_1, k_2, \theta_2) = P_{N_1,N_2}(k_1, k_2, \theta_2)/\Delta f; \; \Delta f = 1/(N_1N_2).$$

Let us estimate the efficiency of increasing the detailing of the two-dimensional energy spectrum $G_{N_1,N_2}(k_1, k_2)$ using 2D DFT-VP in comparison with the classical 2D DFT.

Evaluation of the efficiency of Fourier-processing of signals padded with zero samples in 2D DFT-VP basis

The increase in the detailing of the two-dimensional energy spectrum $G_{N_1,N_2}(k_1, k_2)$ by r_2 times is carried out by padding the horizontal period of the 2D signal $X_{N_1\times N_2}$ with $(r_2 - 1)$ zero matrices $\mathbf{O}_{N_1\times N_2}$ (1). Padding the horizontal period of a 2D signal $X_{N_1\times N_2}$ with $(r_2 - 1)$ zero matrices $\mathbf{O}_{N_1\times N_2}$ makes it possible to obtain a new 2D signal $X_{N_1\times(N_2r_2)}$ from a 2D signal $X_{N_1\times N_2}$.

Applying the 2D DFT in algebraic form to the 2D signal $X_{N_1\times(N_2r_2)}$, we obtain the number of coefficients (bins) of 2D DFT $S_{N_1,N_2r_2}(k_1, k_2)$, which is r_2 times greater than with 2D DFT of the signal $X_{N_1\times N_2}$. However, obtaining a r_2 times more detailed energy spectrum $G_{N_1,N_2r_2}(k_1, k_2)$ by a method based on the separability of the 2D DFT kernel, will require additional $(r_2 - 1)N_1N_2$ cells for storing zero samples and implementing $N_1N_2r_2(N_1 + N_2r_2)$ additional complex operations.

Obtaining an r_2 times detailed energy spectrum $G_{N_1,N_2r_2}(k_1, k_2)$ by a method based on the separability property of the 2D DFT-VP kernel does not require additional RAM (storage) for storing zero samples and requires $N_1N_2r_2(N_1 + N_2)$ complex operations. Thus, the use of 2D DFT-VP instead of the classic 2D DFT allows:

— decrease number of complex operations by $\gamma = \dfrac{N_1 + N_2r_2}{N_1 + N_2}$ times;

— decrease storage size by r_2 times;

— parallelize the process of detailing the two-dimensional energy spectrum $G_{N_1,N_2}(k_1, k_2)$, thus reducing the execution time of the 2D DFT by r_2 times.

Conclusions

Discrete Fourier transforms with a variable parameter have been developed. These transforms make it possible to efficiently process two-dimensional signals, the horizontal periods of which are padded with zero samples. The generalization of classical two-dimensional discrete Fourier transform is based on a new method of splitting the rectangular matrix of discrete Fourier transform into square matrices. The splitting of rectangular matrices into square matrices is carried out by using the ordering of the columns of rectangular matrices using the equivalence relation — the relation of comparability in modulus. The properties of the bases of the proposed transformations are investigated. The validity for Fourier transforms with variable parameters of the following theorems is proved: linearity, shift, correlation, and Parseval's equality.

New concepts of digital spectral Fourier analysis are introduced: the concept of parametric shift of two-dimensional signal and parametric periodicity of two-dimensional signal. The estimation of the reducing the amount of RAM (random access memory) needed and the number of calculations when applying the proposed transforms is carried out in comparison with the application of classical two-dimensional discrete Fourier transform to 2D signals padded with zero samples. Developed two-dimensional discrete Fourier transform with variable parameters can significantly reduce the cost of Fourier-processing of two-dimensional signals (including images), padded with zero samples. In addition, the developed transforms also allow parallelizing the process, thus significantly reducing Fourier-processing time. Note one more application of developed two-dimensional discrete Fourier transform with variable parameters: determination of the parameters of 2D hidden periodicities by varying the parameter θ_2.

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Теоретические основы цифрового векторного фурье-анализа двумерных сигналов, дополненных нулевыми отсчетами

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Введение: практика применения фурье-обработки финитных двумерных сигналов (в том числе изображений), подтвердив ее действенность, выявила и ряд присущих ей отрицательных эффектов. Известный метод борьбы с негативными эффектами фурье-обработки — операция дополнения сигналов нулями. Однако применение этой операции приводит к необходимости обеспечения информационно-управляющих систем дополнительной памятью и проведения ими непроизводительных вычислений. **Цель:** разработка новых дискретных преобразований Фурье для эффективной и результативной обработки двумерных сигналов, дополненных нулями. **Методы:** предложен новый метод разбивки прямоугольной матрицы дискретного преобразования Фурье на квадратные матрицы. Метод основан на применении для упорядочения строк (столбцов) матрицы Фурье отношения сравнимости по модулю. **Результаты:** разработаны новые дискретные преобразования Фурье с варьируемыми параметрами, являющиеся обобщением классического дискретного преобразования Фурье. Исследованы свойства базисов преобразований Фурье с варьируемыми параметрами. Для данных преобразований доказаны теоремы линейности, сдвига, корреляции и равенство Парсеваля. В цифровой спектральный фурье-анализ введены понятия параметрического сдвига двумерного сигнала и параметрической периодичности двумерного сигнала. Выполнена оценка сокращения требуемого объема памяти и числа вычислений в случае применения предложенных преобразований. Проведено их сравнение с дискретным преобразованием Фурье. **Практическая значимость:** разработанные дискретные преобразования Фурье с варьируемыми параметрами позволяют существенно сократить затраты на фурье-обработку двумерных сигналов (в том числе изображений), дополненных нулями.

Ключевые слова — дискретное преобразование Фурье, двумерный сигнал, фурье-обработка, эффекты дискретного преобразования Фурье, базис, варьируемый параметр.

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