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Articles

Some new symmetric Hadamard matrices

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Introduction: *It is conjectured that the symmetric Hadamard matrices of order 4v exist for all odd integers v > 0. In recent years, their existence has been proven for many new orders by using a special method known as the propus construction. This construction uses difference families X_k (k = 1, 2, 3, 4) over the cyclic group $\mathbf{Z}_v$ (integers mod v) with parameters (v; k_1, k_2, k_3, k_4; λ) where X_1 is symmetric, $X_2 = X_3$, and $k_1 + 2k_2 + k_4 = v + \lambda$. It is also conjectured that such difference families (known as propus families) exist for all parameter sets mentioned above excluding the case when all the k_i are equal. This new conjecture has been verified for all odd v ≤ 53.* ***Purpose:*** *To construct many new symmetric Hadamard matrices by using the propus construction and to provide further support for the above-mentioned conjecture.* ***Results:*** *The first examples of symmetric Hadamard matrices of orders 4v are presented for v = 127 and v = 191. The systematic computer search for symmetric Hadamard matrices based on the propus construction has been extended to cover the cases v=55, 57, 59, 61, 63.* ***Practical relevance:*** *Hadamard matrices are used extensively in the problems of error-free coding, and compression and masking of video information.*

Keywords *— symmetric Hadamard matrices, propus construction, propus difference families.*

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Introduction

We fix some notation which will be used throughout this note. Let X_i, $i = 1, 2, 3, 4$, be a difference family (DF) in a finite abelian group G (written additively) and let

$$(v;\, k_1, k_2, k_3, k_4;\, \lambda)$$

be its parameter set (PS). Thus $v = |G|$, $|X_i| = k_i$ and $\Sigma k_i(k_i - 1) = \lambda(v - 1)$, where $|X|$ denotes the cardinality of a finite set X. If $\Sigma k_i = \lambda + v$ we say that this PS is a *Goethals — Seidel parameter set* (GSPS) and that this DF is a *Goethals — Seidel difference family* (GSDF). If the X_i form a GSDF and we replace one of the blocks by its set-theoretic complement in G, we again obtain a GSDF although the parameter set may change. For that reason we shall always assume that all the $k_i \le v/2$.

Each GSDF in G gives a Hadamard matrix $\mathbf{H}$ of order $4v$. For more details about this construction see e.g. [1, 2]. Briefly, each X_i provides a G-invariant matrix $\mathbf{A}_i$ of order v, and $\mathbf{H}$ is obtained by plugging the $\mathbf{A}_i$ into the well known *Goethals — Seidel array*

$$\mathbf{GSA} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2\mathbf{R} & \mathbf{A}_3\mathbf{R} & \mathbf{A}_4\mathbf{R} \\ -\mathbf{A}_2\mathbf{R} & \mathbf{A}_1 & -\mathbf{R}\mathbf{A}_4 & \mathbf{R}\mathbf{A}_3 \\ -\mathbf{A}_3\mathbf{R} & \mathbf{R}\mathbf{A}_4 & \mathbf{A}_1 & -\mathbf{R}\mathbf{A}_2 \\ -\mathbf{A}_4\mathbf{R} & -\mathbf{R}\mathbf{A}_3 & \mathbf{R}\mathbf{A}_2 & \mathbf{A}_1 \end{bmatrix}.$$

We recall that a matrix $\mathbf{A} = (a_{x,y})$ with indices $x, y \in G$ is *G-invariant* if $a_{x+z,y+z} = a_{x,y}$ for all $x, y, z \in G$. The matrix $\mathbf{R} = (r_{x,y})$ may be defined by the formula $r_{x,y} = \delta_{x+y,0}$, $x, y \in G$, where δ is the Kronecker symbol.

For a subset X of G, we say that it is *symmetric* if $-X = X$, and we say that it is *skew* if G is a disjoint union of X, $-X$ and $\{0\}$. If at least one of the blocks X_i of a GSDF is skew then, after rearranging the X_i so to have X_1 skew, the Hadamard matrix $\mathbf{H}$ will be skew Hadamard, i.e. such that $\mathbf{H} + \mathbf{H}^T = 2\mathbf{I}_{4v}$ (T denotes the matrix transposition, and $\mathbf{I}_k$ is the identity matrix of order k).

In order to obtain a symmetric Hadamard matrix $\mathbf{H}$ we require that two of the blocks X_i are the same and that one of the other two blocks is symmetric. A *propus parameter set* (PPS) is a GSPS having $k_i = k_j$ for some $i \ne j$. By permuting the k_i's we may assume that $k_2 = k_3$ and $k_1 \ge k_4$. In that case we say that this PPS is *normalized*. Note that these conditions in general do not specify the k_i's uniquely. For instance the PPSs (5; 1, 2, 2, 1; 1) and (5; 2, 1, 1, 2; 1) are both normalized but they become the same if we ignore the ordering of the k_i's.

We say that a GSDF is a *propus difference family* (PDF) if $X_i = X_j$ for some $i \ne j$ and one of the other two blocks is symmetric. If the X_i's form a PDF then, after rearranging the blocks we may assume that $X_2 = X_3$ and that X_1 is symmetric. Then we plugg the corresponding matrix blocks $\mathbf{A}_i$ into the so called propus array (PA) to construct a symmetric Hadamard matrix of order $4v$. This construction is known as the *propus construction*. It has been first introduced in [3]. For the reader's convenience we display the propus array

$$\mathbf{PA} = \begin{bmatrix} -\mathbf{A}_1 & \mathbf{A}_2\mathbf{R} & \mathbf{A}_3\mathbf{R} & \mathbf{A}_4\mathbf{R} \\ \mathbf{A}_3\mathbf{R} & \mathbf{R}\mathbf{A}_4 & \mathbf{A}_1 & -\mathbf{R}\mathbf{A}_2 \\ \mathbf{A}_2\mathbf{R} & \mathbf{A}_1 & -\mathbf{R}\mathbf{A}_4 & \mathbf{R}\mathbf{A}_3 \\ \mathbf{A}_4\mathbf{R} & -\mathbf{R}\mathbf{A}_3 & \mathbf{R}\mathbf{A}_2 & \mathbf{A}_1 \end{bmatrix}.$$

Note that PA is obtained from GSA by multiplying the first column by –1 and interchanging the second and the third rows.

From now on we assume that $G = \mathbf{Z}_v$, a cyclic group of order v, and that v is odd. Under this assumption, the matrix blocks A_i will be circulants. All PPSs for $v \le 41$ are listed in [4] together with the corresponding PDF's. There was only one case of a PPS having no PDF, namely (25; 10, 10, 10, 10; 15). Similarly, the cases $41 < v \le 51$ were handled in [5], and the case $v = 53$ in [6]. Again there was one exceptional case, (49; 21, 21, 21, 21; 35). In the present note, for each PPS with $51 < v \le 63$ we exhibit at least one PDF. For more information on the exceptional cases see [5].

Symmetric Hadamard matrices of orders 508 and 764

The *symmetry symbol* (abc) written immediately after a PPS shows the symmetry types of the three blocks X_1, X_2 and X_4. More precisely, the letter s means that we require that the corresponding block be symmetric, the letter k is used if we require that block to be skew, and the symbol $*$ is used otherwise. In particular $a = s$ means that we require X_1 to be symmetric, $a = k$ means that we require X_1 to be skew, and $a = *$ means that no symmetry condition is imposed on X_1.

The group of units $\mathbf{Z}_v^*$ acts on $\mathbf{Z}_v$ by multiplication. It may happen that there is a nontrivial subgroup H of $\mathbf{Z}_v^*$ such that some block X_i of a PDF is a union of orbits of H. In such case we may specify X_i by writing it as HY_i, where Y_i is a set of representatives of the H-orbits contained in X_i.

For $v = 127$ we give five nonequivalent PDFs and for $v = 191$ only one.

$v = 127$, $H = \{1, 19, 107\}$

(127; 57, 61, 61, 55; 107) ($**s$)
$X_1 = H\{4, 5, 6, 9, 12, 15, 23, 24, 30, 33, 36, 39, 45, 52, 58, 59, 60, 64, 66\}$
$X_2 = H\{0, 4, 5, 6, 13, 15, 17, 26, 30, 32, 40, 46, 51, 53, 58, 59, 60, 64, 65, 66, 72\}$
$X_4 = H\{0, 2, 4, 8, 9, 12, 15, 23, 24, 26, 30, 33, 40, 46, 51, 52, 53, 65, 71\}$

(127; 60, 60, 60, 54; 107) ($s**$)
$X_1 = H\{1, 5, 6, 11, 13, 15, 16, 17, 20, 23, 24, 29, 32, 45, 46, 52, 58, 66, 71, 72\}$
$X_2 = H\{2, 4, 5, 11, 12, 15, 16, 18, 22, 23, 29, 33, 36, 39, 46, 51, 52, 53, 60, 71\}$
$X_4 = H\{6, 8, 17, 20, 22, 23, 30, 33, 36, 39, 45, 51, 58, 59, 60, 64, 66, 71\}$

(127; 60, 60, 60, 54; 107) ($**s$)
$X_1 = H\{1, 2, 3, 4, 5, 6, 9, 11, 12, 13, 15, 17, 23, 24, 32, 33, 39, 46, 64, 65\}$
$X_2 = H\{2, 5, 9, 10, 12, 13, 15, 16, 17, 29, 33, 36, 39, 40, 45, 51, 53, 58, 60, 66\}$
$X_4 = H\{1, 5, 6, 10, 11, 13, 16, 17, 20, 23, 30, 32, 45, 58, 64, 65, 66, 71\}$

(127; 58, 60, 60, 55; 106) ($**s$)
$X_1 = H\{0, 2, 3, 4, 5, 12, 13, 16, 17, 18, 20, 22, 29, 30, 46, 51, 53, 58, 59, 71\}$
$X_2 = H\{8, 9, 10, 16, 20, 22, 23, 24, 26, 29, 32, 36, 45, 46, 51, 52, 59, 60, 65, 78\}$
$X_4 = H\{0, 3, 5, 10, 11, 17, 18, 22, 24, 29, 32, 39, 45, 52, 58, 59, 60, 64, 72\}$

(127; 60, 57, 57, 58; 105) ($s**$)
$X_1 = H\{2, 4, 6, 10, 12, 13, 15, 17, 23, 24, 26, 36, 40, 46, 51, 52, 58, 64, 71, 78\}$
$X_2 = H\{1, 2, 3, 4, 10, 16, 17, 18, 20, 23, 29, 30, 45, 51, 52, 58, 64, 66, 72\}$
$X_4 = H\{0, 2, 5, 8, 9, 10, 11, 13, 15, 17, 18, 30, 39, 40, 46, 53, 58, 60, 66, 78\}$

$v = 191$, $H = \{1, 39, 49, 109, 184\}$

(191; 91, 90, 90, 85; 165) ($s**$)
$X_1 = H\{0, 1, 3, 4, 7, 9, 16, 17, 18, 21, 22, 28, 31, 36, 57, 61, 62, 68, 112\}$

$X_2 = H\{1, 4, 14, 16, 18, 19, 22, 23, 28, 29, 31, 32, 34, 36, 38, 61, 62, 68\}$
$X_4 = H\{1, 2, 9, 11, 12, 17, 18, 22, 28, 29, 31, 32, 38, 41, 56, 61, 66\}$

Small orders of symmetric Hadamard matrices

There are several known infinite series of PDFs [3, 7]. We shall use only two of them. The first one is essentially the Turyn series [8] with $v = (q+1)/2$, q a prime power $\equiv 1 \pmod 4$, and all four blocks X_i symmetric. The second one is essentially the series constructed in [9] (see also [7]) to which we refer as the XXSW-series. In this case $v = (q+1)/4$, q a prime power $\equiv 3 \pmod 8$, and we may arrange the blocks so that X_1 is skew, $X_2 = X_3$ and X_4 is symmetric.

In the handbook [10] published in 2007 it is indicated (see Table 1.52, p. 277) that, for odd $v < 200$, no symmetric Hadamard matrices of order $4v$ are known for

$$v = 23, 29, 39, 43, 47, 59, 65, 67, 73, 81, 89, 93,$$
$$101, 103, 107, 109, 113, 119, 127, 133, 149, 151, 153, 163, 167, 179, 183, 189, 191, 193$$

The cases $v = 23$ and $v = 81$ should not have been included. For the case $v = 23$ see [7]. For $v = 81$ note that symmetric Hadamard matrices of orders $4 \cdot 9^k$, $k \geq 1$ integer, were constructed by Turyn [11] back in 1984. Moreover, the Bush-type Hadamard matrix of order $4 \cdot 81 = 324$ constructed in 2001 [12] is also symmetric.

The propus construction has been used in several recent papers [3, 4–7, 13] to construct symmetric Hadamard matrices of new orders. By taking into account these results and those from the previous section, the above list of undecided cases reduces to
$v = 65, 89, 93, 101, 107, 119, 133, 149, 153, 163, 167, 179, 183, 189, 193$.

List of PPSs and PDFs for odd v, $53 < v \leq 63$

The following conventions and notation will be used in the listings below. We have $\mathbf{Z}_v = \{0, 1, 2, \ldots, v-1\}$ and recall that v is odd. Let $X \subseteq \mathbf{Z}_v$ and $k = |X|$. Define $X' = X \cap \{1, 2, \ldots, (v-1)/2\}$. In particular $\mathbf{Z}'_v = \{1, 2, \ldots, (v-1)/2\}$.

If X is skew then $k = (v-1)/2$ and

$$X = X' \cup \left(-(\mathbf{Z}'_v \setminus X')\right).$$

If X is symmetric then

$$X = \begin{cases} X' \cup (-X'), & \text{for } k \text{ even;} \\ \{0\} \cup X' \cup (-X'), & \text{for } k \text{ odd.} \end{cases}$$

Hence, a skew X can be recovered uniquely from X'. This is also true for symmetric X provided we know the parity of k.

For a PDF X_i, $i = 1, 2, 3, 4$, with normalized PPS $(v; k_1, k_2, k_3, k_4; \lambda)$ we always assume that $X_2 = X_3$. Thus it suffices to specify only the blocks X_1, X_2 and X_4. We say that a PPS is *exceptional* if all the k_i are equal. The following conjecture is implicit in [4–6]. It has been verified there for odd $v \leq 53$.

Conjecture 1. *For each normalized and non-exceptional PPS* $(v; k_1, k_2, k_3, k_4; \lambda)$ *there exist PDFs with symmetry symbols* $(s{*}{*})$ *and* $({*}{*}s)$.

The list below shows that the conjecture is true also for $v = 55, 57, 59, 61, 63$.

If a block X_i is symmetric or skew, in order to save space we record only X'_i. As the k_i are specified by the PPS, X_i can be recovered uniquely from X'_i.

Example 1. *For the first PDF below, the symmetry symbol* $(s{*}{*})$ *shows that* X_1 *must be symmetric. As* $X'_1 = \{5, 6, 7, 9, 10, 13, 15, 16, 19, 21, 23, 25, 27\}$ *we have* $-X'_1 = \{28, 30, 32, 34, 36, 39, 40, 42, 45, 46, 48, 49, 50\}$. *As* $k_1 = 27$ *is odd we have* $X_1 = \{0\} \cup X'_1 \cup (-X'_1)$.

$$v = 55$$

$(55; 27, 25, 25, 21; 43)$ $(s{*}{*})$
$X'_1 = \{5, 6, 7, 9, 10, 13, 15, 16, 19, 21, 23, 25, 27\}$

X_2 = {0, 1, 2, 3, 4, 6, 9, 10, 14, 17, 19, 24, 26, 29, 30, 34, 37, 38, 39, 40, 41, 47, 48, 52, 53}
X_4 = {0, 3, 4, 5, 10, 11, 12, 14, 16, 17, 18, 19, 21, 22, 24, 30, 36, 43, 44, 46, 47}

(55; 27, 25, 25, 21; 43) (**s)
X_1 = {0, 4, 5, 6, 7, 9, 10, 12, 15, 20, 21, 24, 25, 26, 28, 32, 33, 34, 38, 39, 41, 44, 45, 51, 52, 53, 54}
X_2 = {0, 3 , 5, 7, 8, 9, 10, 19, 23, 25, 28, 31, 32, 33, 34, 36, 37, 40, 41, 43, 44, 45, 47, 48, 53}
X_4' = {2, 4, 6, 9, 12, 13, 18, 19, 20, 23}

(55; 27, 24, 24, 22; 42) (s**)
X_1' = {1, 4, 6, 7, 9, 11, 14, 15, 18, 20, 23, 24, 27}
X_2 = {0, 4, 6, 7, 12, 14, 15, 16, 18, 23, 25, 26, 30, 31, 33, 34, 35, 36, 37, 40, 41, 46, 47, 53}
X_4 = {0, 1, 4, 6, 16, 17, 18, 19, 20, 21, 22, 23, 24, 27, 29, 31, 33, 36, 42, 43, 44, 46}

(55; 27, 24, 24, 22; 42) (**s)
X_1 = {0, 1, 5, 8, 10, 12, 15, 16, 17, 24, 25, 26, 29, 30, 34, 37, 39, 40, 41, 42, 44 ,47, 48, 50, 52, 53, 54}
X_2 = {0, 10, 13, 14, 15, 16, 17, 20, 21, 22, 24, 26, 30, 31, 33, 34, 35, 41, 43, 44, 47, 49, 50, 53}
X_4' = {1, 6, 11, 12, 13, 16, 19, 20, 22, 24, 27}

(55; 26, 23, 23, 24; 41) (s**)
X_1' = {1, 2, 5, 8, 10, 11, 13, 14, 15, 19, 21, 23, 27}
X_2 = {0, 2, 3, 4, 5, 6, 7, 10, 11, 14, 18, 24, 25, 30, 35, 37, 39, 40, 41, 42, 50, 51, 52}
X_4 = {0, 2, 5, 6, 8, 14, 16, 19, 20, 21, 22, 24, 25, 28, 31, 32, 33, 37, 38, 40, 45, 47, 49, 52}

(55; 26, 23, 23, 24; 41) (**s)
X_1 = {0, 2, 7, 11, 14, 15, 17, 18, 23, 24, 25, 28, 29, 30, 31, 36, 37, 38, 39, 42, 44, 45, 46, 48, 50, 53}
X_2 = {0, 3, 8, 12, 13, 14, 18, 19, 23, 26, 33, 34, 36, 38, 42, 45, 46, 47, 48, 49, 50, 51, 52}
X_4' = {1, 2, 7, 10, 11, 13, 17, 19, 21, 24, 26, 27}

(55; 24, 27, 27, 21; 44) (s**)
X_1' = {6, 8, 10, 13, 15, 16, 18, 19, 20, 22, 25, 26}
X_2 = {0, 2, 4, 5, 8, 13, 14, 16, 17, 19, 25, 26, 27, 32, 33, 34, 37, 38, 39, 40, 41, 42, 44, 49, 50, 53, 54}
X_4 = {0, 4, 5, 8, 11, 12, 13, 16, 18, 20, 22, 24, 31, 33, 34, 36, 37, 41, 42, 44, 51}

(55; 24, 27, 27, 21; 44) (**s)
X_1 = {0, 3, 11, 12, 13, 14, 16, 17, 23, 24, 26, 27, 29, 30, 35, 38, 39, 40, 41, 42, 47, 48, 49, 50}
X_2 = {0, 1, 3, 4, 5, 6, 8, 9, 14, 16, 18, 21, 22, 27, 28, 32, 35, 36, 37, 39, 43, 47, 49, 51, 52, 53, 54}
X_4' = {3, 5, 9, 10, 14, 16, 17, 20, 23, 25}

(55; 24, 25, 25, 22; 41) (s**)
X_1' = {1, 3, 5, 6, 7, 11, 14, 15, 16, 18, 21, 25}
X_2 = {0, 9, 11, 12, 14, 17, 20, 22, 23, 24, 25, 26, 27, 30, 31, 33, 37, 38, 42, 46, 47, 48, 49, 52, 54}
X_4 = {0, 4, 5, 7, 8, 11, 12, 16, 18, 19, 21, 24, 25, 26, 28, 34, 35, 39, 41, 43, 53, 54}

(55; 24, 25, 25, 22; 41) (**s)
X_1 = {0, 2, 3, 5, 13, 14, 16, 17, 21, 22, 23, 26, 29, 32, 37, 38, 42, 43, 44, 45, 46, 48, 49, 51}
X_2 = {0, 1, 2, 3, 4, 10, 11, 15, 17, 18, 19, 21, 22, 24, 25, 27, 28, 30, 32, 34, 35, 39, 40, 44, 46}
X_4' = {2, 3, 5, 7, 10, 11, 15, 21, 23, 25, 26}

(55; 23, 26, 26, 22; 42) (sss), Turyn series
X_1' = {6, 7, 10, 11, 15, 17, 18, 19, 21, 24, 26}
X_2' = {1, 2, 4, 8, 14, 16, 17, 18, 19, 23, 24, 25, 27}
X_4' = {6, 7, 10, 11, 15, 17, 18, 19, 21, 24, 26}

$v = 57$, H = {1, 7, 49}

(57; 28, 28, 28, 21; 48) (s**)
X_1' = {1, 2, 6, 8, 10, 11, 14, 16, 17, 18, 21, 22, 25, 27}
X_2 = {0, 2, 3, 5, 6, 7, 11, 12, 13, 15, 18, 23, 24, 25, 26, 27, 28, 29, 30, 32, 34, 39, 40, 42, 46, 49, 50, 55}
X_4 = {0, 1, 2, 3, 5, 6, 8, 9, 12, 13, 14, 15, 19, 23, 30, 31, 32, 39, 45, 48, 51}

(57; 28, 28, 28, 21; 48) ($k{*}s$), XXSW series
$X_1' = \{2, 4, 12, 13, 15, 21, 23, 24, 25, 27, 28\}$
$X_2 = \{1, 3, 5, 8, 9, 12, 15, 20, 23, 24, 26, 27, 29, 31, 32, 33, 35, 36, 37, 41, 42, 45, 49, 50, 51, 52, 53, 55\}$
$X_4' = \{1, 4, 6, 13, 14, 15, 19, 20, 21, 26\}$

(57; 27, 26, 26, 22; 44) ($s{*}{*}$), all X_i are H-invariant
$X_1' = \{3, 6, 9, 10, 11, 13, 15, 19, 20, 21, 23, 24, 26\}$
$X_2 = \{4, 6, 8, 9, 15, 16, 19, 23, 25, 28, 30, 31, 37, 38, 39, 42, 43, 44, 45, 46, 47, 48, 50, 51, 55, 56\}$
$X_4 = \{2, 4, 6, 9, 10, 11, 13, 14, 20, 22, 25, 26, 28, 30, 34, 38, 39, 40, 41, 42, 45, 52\}$

(57; 27, 26, 26, 22; 44) (${*}{*}s$)
$X_1 = \{0, 8, 9, 10, 12, 13, 15, 18, 19, 20, 25, 30, 33, 34, 37, 40, 41, 43, 47, 48, 49, 51, 52, 53, 54, 55, 56\}$
$X_2 = \{0, 4, 8, 9, 11, 16, 17, 18, 22, 24, 25, 26, 27, 28, 30, 31, 33, 34, 37, 38, 42, 43, 51, 53, 54, 55\}$
$X_4' = \{2, 4, 7, 9, 11, 12, 14, 17, 21, 23, 28\}$

(57; 27, 25, 25, 23; 43) ($s{*}{*}$), all X_i are H-invariant
$X_1' = \{3, 6, 9, 10, 11, 13, 15, 19, 20, 21, 23, 24, 26\}$
$X_2 = \{2, 3, 4, 14, 16, 21, 22, 24, 25, 28, 29, 30, 32, 33, 36, 38, 39, 40, 41, 43, 45, 52, 53, 54, 55\}$
$X_4 = \{1, 2, 5, 7, 10, 13, 14, 17, 19, 24, 29, 30, 32, 34, 35, 36, 38, 39, 41, 45, 49, 53, 54\}$

(57; 27, 25, 25, 23; 43) (${*}{*}s$)
$X_1 = \{0, 1, 7, 11, 15, 16, 17, 19, 20, 21, 24, 25, 26, 28, 30, 35, 36, 37, 38, 40, 44, 46, 47, 48, 49, 51, 54\}$
$X_2 = \{0, 1, 2, 3, 7, 10, 14, 17, 19, 20, 23, 26, 31, 32, 34, 36, 37, 38, 41, 42, 43, 44, 45, 46, 49\}$
$X_4' = \{2, 4, 6, 9, 10, 11, 15, 16, 18, 23, 26\}$

(57; 25, 25, 25, 24; 42) (sss), Turyn series
$X_1' = \{2, 3, 8, 9, 10, 18, 20, 22, 23, 24, 26, 27\}$
$X_2' = \{6, 7, 9, 10, 14, 16, 19, 21, 24, 25, 27, 28\}$
$X_4' = \{2, 3, 8, 9, 10, 18, 20, 22, 23, 24, 26, 27\}$

$v = 59$

The third and the sixth PDF below are taken from **[6]**.

(59; 28, 29, 29, 22; 49) ($s{*}{*}$)
$X_1' = \{1, 3, 5, 8, 10, 11, 13, 15, 16, 20, 21, 22, 26, 29\}$
$X_2 = \{0, 1, 3, 7, 8, 9, 10, 12, 13, 15, 16, 19, 21, 22, 25, 27, 29, 34, 35, 36, 37, 38, 39, 40, 44, 51, 54, 55, 58\}$
$X_4 = \{0, 3, 4, 5, 7, 8, 14, 15, 16, 17, 19, 22, 24, 27, 28, 33, 39, 49, 53, 54, 55, 56\}$

(59; 28, 29, 29, 22; 49) (${*}{*}s$)
$X_1 = \{0, 5, 6, 7, 8, 9, 10, 12, 17, 18, 20, 25, 26, 27, 34, 35, 39, 42, 44, 45, 47, 48, 49, 50, 51, 54, 55, 58\}$
$X_2 = \{0, 1, 2, 3, 4, 5, 6, 9, 11, 13, 14, 18, 20, 24, 25, 26, 29, 31, 32, 35, 37, 44, 45, 47, 48, 51, 55, 56, 58\}$
$X_4' = \{1, 5, 7, 11, 15, 16, 18, 20, 21, 23, 29\}$

(59; 27, 25, 25, 26; 44) ($s{*}{*}$)
$X_1' = \{2, 4, 7, 8, 12, 13, 15, 16, 17, 18, 20, 23, 29\}$
$X_2 = \{1, 2, 4, 5, 12, 13, 17, 19, 20, 21, 22, 23, 26, 27, 31, 35, 37, 38, 40, 44, 47, 49, 50, 55, 57\}$
$X_4 = \{3, 7, 12, 13, 14, 16, 18, 19, 20, 22, 23, 24, 25, 26, 31, 32, 33, 34, 36, 38, 43, 45, 46, 50, 51, 53\}$

(59; 27, 25, 25, 26; 44) (${*}{*}s$)
$X_1 = \{0, 1, 3, 5, 6, 7, 8, 9, 12, 15, 18, 20, 28, 29, 31, 33, 34, 35, 38, 42, 44, 47, 48, 49, 55, 56, 58\}$
$X_2 = \{0, 3, 4, 5, 7, 10, 16, 21, 22, 24, 25, 26, 28, 29, 32, 33, 34, 38, 39, 40, 41, 43, 48, 49, 52\}$
$X_4' = \{1, 3, 5, 7, 10, 12, 13, 15, 18, 19, 20, 27, 28\}$

(59; 26, 28, 28, 23; 46) ($s{*}{*}$)
$X_1' = \{4, 6, 10, 12, 13, 15, 17, 21, 22, 24, 25, 27, 29\}$
$X_2 = \{0, 1, 4, 5, 6, 7, 10, 11, 12, 13, 14, 17, 21, 22, 23, 27, 33, 34, 36, 37, 39, 41, 42, 45, 52, 55, 56, 57\}$
$X_4 = \{0, 7, 9, 12, 13, 14, 22, 25, 26, 27, 28, 31, 33, 35, 39, 40, 46, 47, 49, 51, 55, 57, 58\}$

(59; 26, 28, 28, 23; 46) (**s)
X_1 = {2, 3, 10, 12, 13, 14, 16, 18, 19, 26, 28, 29, 36, 38, 39, 40, 42, 44, 46, 47, 49, 50, 53, 54, 55, 57}
X_2 = {4, 5, 7, 11, 12, 16, 17, 24, 25, 26, 27, 28, 29, 33, 34, 37, 39, 40, 42, 43, 44, 45, 47, 49, 51, 53, 56, 58}
X_4' = {1, 4, 5, 7, 8, 11, 14, 20, 25, 28, 29}

$v = 61$, H_1 = {1, 13, 47}, H_2 = {1, 9, 20, 34, 58}

(61; 30, 29, 29, 23; 50) (s**)
X_1' = {2, 3, 8, 10, 11, 13, 14, 16, 18, 20, 22, 23, 24, 28, 30}
X_2 = {0, 1, 2, 4, 9, 10, 11, 13, 18, 19, 22, 24, 26, 27, 37, 38, 39, 40, 42, 43, 44, 48, 49, 51, 52, 55, 56, 58, 59}
X_4 = {0, 6, 7, 8, 11, 12, 14, 16, 17, 23, 26, 30, 31, 33, 34, 35, 37, 38, 40, 43, 45, 49, 50}

(61; 30, 29, 29, 23; 50) (**s)
X_1 = {1, 2, 5, 6, 7, 8, 11, 12, 14, 15, 19, 20, 21, 22, 24, 27, 28, 30, 31, 32, 38, 39, 40, 41, 43, 45, 46, 55, 58, 60}
X_2 = {0, 2, 4, 8, 9, 10, 13, 14, 17, 18, 20, 23, 24, 25, 26, 27, 32, 37, 38, 44, 45, 47, 49, 53, 55, 56, 58, 59, 60}
X_4' = {2, 10, 13, 15, 17, 18, 19, 21, 22, 26, 29}

(61; 30, 26, 26, 26; 47) (s**), all X_i are H_2-invariant
X_1' = {1, 3, 4, 5, 9, 12, 13, 14, 15, 16, 19, 20, 22, 25, 27}
X_2 = {0, 1, 6, 8, 9, 11, 12, 20, 21, 25, 26, 28, 30, 32, 34, 37, 38, 42, 43, 44, 47, 51, 54, 57, 58, 59}
X_4 = {0, 3, 5, 6, 21, 23, 24, 26, 27, 30, 32, 33, 39, 41, 43, 44, 45, 46, 48, 50, 51, 52, 53, 54, 59, 60}

(61; 30, 26, 26, 26; 47) (**s)
X_1 = {0, 2, 4, 5, 6, 8, 9, 10, 12, 18, 19, 21, 22, 23, 25, 26, 27, 28, 30, 32, 34, 36, 37, 42, 49, 55, 56, 57, 58, 59}
X_2 = {0, 2, 5, 8, 9, 10, 15, 18, 24, 27, 28, 31, 33, 35, 38, 39, 44, 45, 46, 47, 49, 50, 51, 52, 59, 60}
X_4' = {2, 3, 6, 9, 11, 14, 18, 19, 21, 23, 24, 25, 29}

(61; 30, 25, 25, 30; 49) (sss), Turyn series
X_1' = {1, 2, 6, 8, 9, 12, 13, 14, 15, 16, 17, 19, 24, 25, 28}
X_2' = {1, 5, 6, 8, 10, 11, 12, 14, 20, 24, 27, 29}
X_4' = {3, 4, 5, 7, 10, 11, 18, 20, 21, 22, 23, 26, 27, 29, 30}

(61; 30, 25, 25, 30; 49) (k*s), XXSW series
X_1' = {3, 4, 6, 13, 14, 15, 16, 19, 21, 22, 23, 24, 26, 27, 30}
X_2 = {5, 6, 8, 12, 14, 15, 17, 20, 31, 32, 33, 36, 40, 44, 45, 46, 48, 49, 51, 53, 54, 55, 56, 59, 60}
X_4' = {1, 3, 5, 9, 10, 13, 15, 16, 17, 20, 22, 26, 27, 29, 30}

(61; 28, 28, 28, 24; 47) (s**)
X_1' = {1, 4, 6, 7, 8, 10, 11, 14, 19, 20, 22, 26, 28, 30}
X_2 = {0, 1, 6, 10, 12, 13, 17, 21, 22, 23, 26, 27, 29, 32, 34, 35, 36, 39, 40, 42, 43, 50, 52, 54, 57, 58, 59, 60}
X_4 = {0, 5, 6, 8, 18, 23, 24, 28, 32, 34, 37, 42, 43, 44, 48, 49, 50, 51, 52, 54, 55, 56, 57, 58}

(61; 28, 28, 28, 24; 47) (**s), all X_i are H_1-invariant
X_1 = {0, 1, 3, 4, 5, 9, 12, 13, 14, 15, 18, 19, 27, 32, 34, 39, 40, 46, 47, 48, 49, 50, 51, 52, 53, 56, 57, 60}
X_2 = {0, 1, 2, 4, 5, 7, 11, 13, 14, 21, 22, 24, 26, 29, 30, 31, 33, 36, 37, 41, 42, 45, 47, 48, 52, 54, 58, 60}
X_4' = {1, 3, 11, 13, 14, 16, 19, 20, 21, 22, 25, 29}

(61; 28, 27, 27, 25; 46) (s**)
X_1' = {1, 2, 3, 4, 5, 7, 9, 11, 17, 18, 20, 24, 27, 29}
X_2 = {0, 4, 6, 7, 11, 12, 15, 18, 19, 21, 26, 30, 31, 32, 33, 35, 36, 41, 43, 44, 45, 48, 49, 51, 52, 53, 54}
X_4 = {0, 4, 10, 11, 13, 16, 17, 21, 22, 27, 30, 32, 37, 38, 40, 42, 43, 44, 46, 47, 51, 53, 54, 55, 56}

(61; 28, 27, 27, 25; 46) (**s), all X_i are H_1-invariant
X_1 = {0, 7, 11, 18, 21, 22, 23, 24, 28, 29, 30, 31, 32, 35, 36, 37, 40, 41, 42, 44, 45, 50, 51, 53, 54, 55, 58, 59}
X_2 = {1, 3, 7, 8, 9, 10, 13, 19, 23, 24, 27, 28, 30, 31, 35, 37, 39, 43, 44, 46, 47, 49, 54, 55, 56, 57, 59}
X_4' = {1, 3, 8, 10, 13, 14, 16, 18, 19, 20, 22, 25}

(61; 25, 30, 30, 25; 49) (s**), all X_i are H_1-invariant
X_1' = {1, 3, 8, 10, 13, 14, 16, 18, 19, 20, 22, 25}

X_2 = {2, 6, 7, 14, 17, 18, 22, 23, 24, 26, 27, 28, 30, 33, 35, 36, 38, 41, 42, 44, 45, 46, 48, 49, 51, 53, 55, 58, 59, 60}
X_4 = {0, 1, 2, 3, 6, 13, 14, 17, 19, 26, 27, 31, 32, 33, 37, 38, 39, 40, 46, 47, 48, 49, 50, 54, 60}

$v = 63$, H_1 = {1, 4, 16}, H_2 = {1, 25, 58}

(63; 31, 26, 26, 30; 50) (*sss*), Turyn series, all X_i are H_2-invariant
X_1' = {1, 3, 4, 5, 7, 12, 14, 15, 19, 20, 25, 26, 28, 29, 31}
X_2' = {7, 8, 9, 11, 14, 18, 19, 21, 23, 27, 28, 29, 31}
X_4' = {1, 3, 4, 5, 7, 12, 14, 15, 19, 20, 25, 26, 28, 29, 31}

(63; 30, 30, 30, 24; 51) (*s***), all X_i are H_1-invariant
X_1' = {1, 2, 4, 8, 9, 11, 13, 16, 18, 19, 22, 25, 26, 27, 31}
X_2 = {7, 9, 11, 13, 14, 15, 18, 19, 22, 25, 26, 27, 28, 30, 35, 36, 37, 38, 39, 41, 44, 45, 49, 50, 51, 52, 54, 56, 57, 60}
X_4 = {5, 9, 13, 15, 17, 18, 19, 20, 22, 23, 25, 26, 29, 31, 36, 37, 38, 41, 51, 52, 53, 55, 60, 61}

(63; 30, 30, 30, 24; 51) (***s*), all X_i are H_1-invariant
X_1 = {3, 5, 6, 9, 10, 12, 13, 14, 17, 18, 19, 20, 23, 24, 26, 29, 30, 33, 34, 35, 36, 38, 39, 40, 41, 48, 52, 53, 56, 57}
X_2 = {3, 5, 7, 9, 10, 12, 13, 15, 17, 18, 19, 20, 23, 26, 27, 28, 29, 34, 36, 38, 40, 41, 45, 48, 49, 51, 52, 53, 54, 60}
X_4' = {5, 7, 9, 10, 14, 17, 18, 20, 23, 27, 28, 29}

(63; 30, 27, 27, 27; 48) (*s***), all X_i are H_2-invariant
X_1' = {1, 5, 8, 9, 11, 16, 17, 18, 19, 22, 23, 25, 27, 29, 31}
X_2 = {3, 4, 7, 12, 15, 16, 17, 20, 22, 26, 27, 28, 29, 32, 37, 41, 43, 44, 45, 46, 47, 48, 49, 51, 54, 59, 60}
X_4 = {4, 6, 9, 10, 13, 17, 18, 19, 24, 27, 29, 31, 32, 33, 34, 36, 37, 40, 41, 43, 44, 45, 47, 52, 54, 55, 61}

(63; 30, 27, 27, 27; 48) (***s*), all X_i are H_1-invariant
X_1 = {3, 6, 11, 12, 13, 19, 22, 23, 24, 25, 26, 27, 29, 30, 33, 37, 38, 39, 41, 43, 44, 45, 46, 48, 50, 52, 53, 54, 57, 58}
X_2 = {3, 11, 12, 13, 14, 15, 19, 22, 25, 26, 31, 35, 37, 38, 41, 43, 44, 46, 48, 50, 51, 52, 55, 56, 58, 60, 61}
X_4' = {1, 4, 7, 9, 14, 16, 18, 21, 22, 25, 26, 27, 28}

(63; 29, 31, 31, 24; 52) (*s***)
X_1' = {2, 3, 5, 8, 9, 10, 12, 15, 16, 22, 24, 26, 28, 31}
X_2 = {0, 1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 15, 20, 23, 24, 28, 29, 30, 31, 34, 40, 42, 43, 45, 49, 50, 52, 58, 59, 60, 61}
X_4 = {0, 2, 8, 9, 10, 12, 15, 16, 20, 25, 26, 30, 33, 34, 37, 39, 45, 46, 48, 50, 57, 60, 61, 62}

(63; 29, 31, 31, 24; 52) (***s*)
X_1 = {1, 2, 3, 11, 15, 18, 19, 21, 22, 26, 27, 28, 29, 30, 33, 34, 35, 36, 38, 39, 45, 48, 49, 51, 52, 55, 59, 60, 61}
X_2 = {0, 2, 3, 5, 10, 11, 17, 18, 19, 21, 23, 24, 28, 30, 32, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 48, 51, 52, 53, 56, 62}
X_4' = {2, 6, 10, 11, 12, 14, 17, 20, 22, 25, 27, 29}

(63; 27, 31, 31, 25; 51) (*s***), all X_i are H_1-invariant
X_1' = {2, 3, 7, 8, 10, 12, 14, 15, 21, 23, 28, 29, 31}
X_2 = {3, 6, 7, 11, 12, 13, 14, 19, 22, 23, 24, 25, 26, 28, 29, 33, 35, 37, 38, 41, 42, 43, 44, 46, 48, 49, 50, 52, 53, 56, 58}
X_4 = {3, 5, 12, 13, 15, 17, 19, 20, 22, 23, 25, 26, 29, 30, 37, 38, 39, 41, 42, 48, 51, 52, 53, 57, 60}

(63; 27, 31, 31, 25; 51) (***s*)
X_1 = {0, 1, 2, 3, 4, 6, 7, 10, 11, 12, 15, 19, 23, 25, 28, 31, 32, 39, 40, 47, 49, 51, 52, 53, 58, 59, 62}
X_2 = {0, 1, 6, 7, 9, 10, 13, 14, 16, 18, 19, 23, 24, 28, 30, 35, 37, 38, 40, 41, 43, 44, 46, 47, 48, 49, 50, 54, 59, 61, 62}
X_4' = {1, 2, 4, 6, 10, 14, 16, 17, 19, 21, 27, 28}

(63; 27, 29, 29, 26; 48) (*s***), all X_i are H_2-invariant
X_1' = {1, 2, 3, 5, 10, 12, 13, 15, 19, 21, 25, 29, 31}

X_2 = {2, 7, 9, 10, 13, 14, 18, 20, 21, 26, 27, 28, 29, 32, 35, 36, 40, 42, 44, 45, 49, 50, 52, 53, 54, 55, 56, 59, 61}
X_4 = {4, 7, 8, 9, 11, 16, 17, 18, 20, 21, 22, 23, 26, 28, 29, 32, 36, 37, 41, 42, 43, 44, 46, 47, 49, 59}

(63; 27, 29, 29, 26; 48) (***s*)
X_1 = {0, 1, 3, 5, 6, 8, 9, 12, 15, 17, 21, 25, 26, 27, 28, 29, 31, 35, 36, 41, 43, 46, 48, 53, 54, 57, 61}
X_2 = {0, 2, 3, 5, 9, 10, 21, 25, 26, 27, 28, 30, 31, 33, 34, 35, 41, 42, 43, 44, 45, 46, 49, 53, 54, 55, 57, 59, 60}
X_4' = {1, 3, 4, 10, 11, 14, 16, 18, 20, 23, 26, 27, 31}

In the case $v = 57$ our list contains two PDFs having different parameter sets and sharing the same symmetric block. The same is true for $v = 61$.

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References

1. Dragomir Ž. Đoković, Ilias S. Kotsireas. Computational methods for difference families in finite abelian groups. *Spec. Matrices*, 2019, vol. 7, pp. 127–141.
2. Seberry J., Yamada M. *Hadamard matrices, sequences, and block designs*. In: *Contemporary design theory*. A Collection of Surveys. D. J. Stinson and J. Dinitz (eds.). John Wiley and Sons, New York, 1992. Pp. 431–560.
3. Seberry J., and Balonin N. A. Two infinite families of symmetric Hadamard matrices. *Australasian Journal of Combinatorics*, 2017, vol. 69(3), pp. 349–357.
4. Balonin N. A., Balonin Y. N., Đoković D. Ž., Karbovskiy D. A., and Sergeev M. B. Construction of symmetric Hadamard matrices. *Informatsionno-upravliaiushchie sistemy* [Information and Control Systems], 2017, no. 5, pp. 2–11. doi:10.15217/issn1684-8853.2017.5.2
5. Balonin N. A., Đoković D. Ž., and Karbovskiy D. A. Construction of symmetric Hadamard matrices of order 4*v* for *v* = 47, 73, 113. *Spec. Matrices*, 2018, vol. 6, no. 1, pp. 11–22. doi.org/10.1515/spma-2018-0002
6. Abuzin L. V., Balonin N. A., Đoković D. Ž., Kotsireas I. S. Hadamard matrices from Goethals — Seidel difference families with a repeated block. *Informatsionno-upravliaiushchie sistemy* [Information and Control Systems], 2019, no. 5, pp. 2–9. doi:10.31799/1684-88532019-5-2-9
7. Mateo O. Di, Đoković D. Ž., Kotsireas I. S. Symmetric Hadamard matrices of order 116 and 172 exist. *Spec. Matrices*, 2015, vol. 3, pp. 227–234.
8. Turyn R. J. An infinite class of Williamson matrices. *J. Combinatorial Theory Ser. A*, 1972, no. 12, pp. 319–321.
9. Xia M., Xia T., Seberry J., and Wu J. An infinite series of Goethals — Seidel arrays. *Discrete Applied Mathematics*, 2005, vol. 145, pp. 498–504.
10. Craigen R., and Kharaghani H. *Hadamard matrices and Hadamard designs*. In: *Handbook of Combinatorial Designs*. 2nd ed. C. J. Colbourn, J. H. Dinitz (eds). Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC Press, Boca Raton, FL, 2007. Pp. 273–280.
11. Turyn R. J. An infinite class of Williamson matrices. *J. Combinatorial Theory Ser. A*, 1972, no. 12, pp. 319–321.
12. Janko Z., Kharaghani H., Tonchev V. The existence of a Bush-type Hadamard matrix of order 324 and two new infinite classes of symmetric designs. *Designs, Codes and Cryptography*, 2001, vol. 24, iss. 2, pp. 225–232.
13. Balonin N. A., and Đoković D. Ž. Symmetric Hadamard matrices of orders 268, 412, 436 and 604. *Informatsionno-upravliaiushchie sistemy* [Information and Control Systems], 2018, no. 4, pp. 2–8. doi:10.31799/1684-8853-2018-4-2-8

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Некоторые новые симметричные матрицы Адамара

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Введение: предполагается, что симметричные матрицы Адамара порядка $4v$ существуют для всех нечетных целых чисел $v > 0$. В последние годы их наличие было доказано для многих новых порядков с помощью специального метода, известного как конструкция пропус. В этой конструкции используются разностные семейства X_k ($k = 1, 2, 3, 4$) над циклической группой $\mathbf{Z}_v$ (целые числа по модулю v) с параметрами (v; k_1, k_2, k_3, k_4; λ), где X_1 симметрично, $X_2 = X_3$ и $k_1 + 2k_2 + k_4 = v + \lambda$. Также предполагается, что такие разностные семейства (известные как пропус-семейства) существуют для всех наборов параметров, упомянутых выше, за исключением случая, когда все k_i равны. Эта новая гипотеза была проверена для всех нечетных $v \le 53$. **Цель:** построить новые симметричные матрицы Адамара, используя конструкцию пропус, и обеспечить дальнейшее подтверждение вышеупомянутой гипотезы. **Результаты:** представлены первые примеры симметричных матриц Адамара порядков $4v$ для $v = 127$ и $v = 191$. Систематический компьютерный поиск симметричных матриц Адамара, основанный на конструкции пропус, был расширен на случаи $v = 55, 57, 59, 61, 63$. **Практическая значимость:** матрицы Адамара широко используются в задачах безошибочного кодирования и сжатия и маскирования видеоинформации.

Ключевые слова — симметричные матрицы Адамара, конструкция пропус, разностные семейства пропусов.

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