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Articles

Exact solution method for Fredholm integro-differential equations with multipoint and integral boundary conditions. Part 2. Decomposition-extension method for squared operators

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Introduction: *In Part 1 of this article, a direct method was presented for examining the solvability and uniqueness problem, and for obtaining a closed-form solution of boundary value problems which incorporate an m^{th} order linear ordinary Fredholm integro-differential operator, or a differential operator, along with multipoint and integral boundary conditions. Here, we focus on a special class of boundary value problems including the composite square of an integro-differential operator and the corresponding non-local boundary conditions.* ***Purpose:*** *To investigate the construction of the unique solution of $2m^{th}$ order boundary value problems in the special case of an operator which can be presented as composite squares of lower m^{th} order ones, and to develop an algorithm for constructing an exact solution for this special case.* ***Results:*** *By decomposition and applying the extension method explicated in Part 1, we provide a formula for obtaining an exact solution of boundary value problems for squared integro-differential operators, or differential operators, with multipoint and integral boundary conditions. This method is simple to use and can be easily incorporated to any Computer Algebra System.*

Keywords *— differential and Fredholm integro-differential equations, multipoint and non-local integral boundary conditions, decomposition of operators, correct operators, exact solutions.*

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Introduction

In the article [1], we presented the development, the applications and the necessity for studying boundary values problems encompassing m^{th} order linear ordinary Fredholm integro-differential operators and general nonlocal boundary conditions such as multipoint and integral boundary conditions. We proposed a direct constructive method for examining the existence and uniqueness of the solution and obtaining it in closed-form. The method was based on the extension theory of linear operators in Banach spaces, in particular on the technique developed in [2] and [3] for solving exactly linear and nonlinear, respectively, integro-differential equations subject to initial and classical boundary conditions.

In this paper, which is a sequel to [1], we study separately a specific type of boundary value problems involving the composite square of an m^{th} order linear ordinary Fredholm integro-differential, or differential, operator, and analogous multipoint and integral boundary conditions. We establish the requirements under which there exists a unique solution and show how to construct it in closed-form by decomposing and utilizing the extension method described in [1].

The decomposition, or factorization, method for problems embracing integro-differential operators and unperturbed conventional boundary conditions is studied in [4]. Therefore, the current work can be seen also as an advancement of [4] where perturbed boundary conditions are considered. Factorization techniques find applications in several areas in sciences and engineering, see, for example, in [5, 6].

The organization of the paper is as follows. We first describe the decomposition-extension method and then we apply the method to solve second and fourth-order differential and integro-differential problems which can be formulated as composite squares of first and second-order problems, respectively. Lastly, some conclusions are quoted.

Decomposition-extension method

Let X be a complex Banach space, usually $X = \mathbb{C}[a, b]$ (or $X = L_p(a, b)$, $p \geq 1$), and $A: X \xrightarrow{on} X$ an m^{th} order linear ordinary differential operator, namely:

$$Au = a_0 u^{(m)} + a_1 u^{(m-1)} + \ldots + a_m u, \tag{1}$$

where $a_i \in \mathbb{R}$ and $u = u(x) \in X_A^m$, where $X_A^m = \mathbb{C}^m[a, b]$ (or $X_A^m = \mathbf{W}_p^m(a, b)$). Let the space $\ker A$ be finite dimensional and $\mathbf{z} = (z_1, \ldots, z_m)$ be a basis of it. Let $\hat{A}$ be a correct restriction of A, specifically $\hat{A}u = Au$ for all u in

$$\mathfrak{D}(\hat{A}) = \{u \in \mathfrak{D}(A) : \mathbf{\Phi}(u) = \mathbf{0}\}, \tag{2}$$

where $\mathbf{\Phi} = \mathrm{col}(\Phi_1, \ldots, \Phi_m)$ is a vector of m bounded linear functionals on X_A^m, which are biorthogonal to $z_1, \ldots, z_m$ and describe some boundary conditions.

Consider the integro-differential operator B: $X \to X$:

$$Bu = Au - \mathbf{gF}(Au),$$
$$\mathfrak{D}(B) = \{u \in \mathfrak{D}(A): \mathbf{\Phi}(u) = \mathbf{N\Psi}(u)\}, \tag{3}$$

and the more complex integro-differential operator B_1: $X \to X$:

$$B_1 u = A^2 u - \mathbf{qF}(Au) - \mathbf{gF}(A^2 u),$$
$$\mathfrak{D}(B_1) = \{u \in \mathfrak{D}(A^2): \mathbf{\Phi}(u) = \mathbf{N\Psi}(u),$$
$$\mathbf{\Phi}(Au) = \mathbf{DF}(Au) + \mathbf{N\Psi}(Au)\}, \tag{4}$$

where A^2 is meant to be the composite product $A^2 = A(A)$, $\mathbf{\Psi} = \mathrm{col}(\Psi_1, \ldots, \Psi_n)$ is a vector of n bounded linear functionals on X_A^m, $\mathbf{F} = \mathrm{col}(F_1, \ldots, F_n)$ is a vector of n bounded linear functionals on X, $\mathbf{g} = (g_1, \ldots, g_n)$, $\mathbf{q} = (q_1, \ldots, q_n) \in X^n$, $q_1, \ldots, q_n$ are linearly independent vectors, and $\mathbf{D}$, $\mathbf{N}$ are $m \times n$ constant matrices. The equations $\mathbf{\Phi}(u) = \mathbf{N\Psi}(u)$ and $\mathbf{\Phi}(Au) = \mathbf{DF}(Au) + \mathbf{N\Psi}(Au)$ symbolize general boundary conditions including multipoint and integral boundary conditions.

The boundary value problems $Bu = f$ and $B_1 u = f$, for any $f \in X$, were studied and solved exactly by utilizing the extension method in [1].

We contemplate here the special case of the boundary value problem $B_1 = f$, $\forall f \in X$, when $B_1 = B^2$; B^2 is understood to be the composite product: $B^2 = B(B)$. For this case we prove the following theorem which provides solvability conditions and describes the decomposition-extension procedure for obtaining the solution in closed form.

Theorem. (i) The operator B_1 is decomposed in $B_1 = B^2$ in the case if

$$\mathbf{g} \in \mathfrak{D}(A)^n,\ \mathbf{q} = A\mathbf{g} - \mathbf{gF}(A\mathbf{g}),\ \mathbf{D} = \mathbf{\Phi}(\mathbf{g}) - \mathbf{N\Psi}(\mathbf{g}). \tag{5}$$

The operator B^2 is defined by

$$B^2 u = A^2 u - [A\mathbf{g} - \mathbf{gF}(A\mathbf{g})]\mathbf{F}(Au) - \mathbf{gF}(A^2 u),$$
$$\mathfrak{D}(B^2) = \{u \in \mathfrak{D}(A^2): \mathbf{\Phi}(u) = \mathbf{N\Psi}(u),$$
$$\mathbf{\Phi}(Au) = [\mathbf{\Phi}(\mathbf{g}) - \mathbf{N\Psi}(\mathbf{g})]\mathbf{F}(Au) + \mathbf{N\Psi}(Au)\}. \tag{6}$$

(ii) If the vectors $\mathbf{q}$, $\mathbf{g}$ and matrices $\mathbf{D}$, $\mathbf{N}$ satisfy (5), then the operator B_1 is injective if and only if

$$\det\mathbf{V} = \det[\mathbf{I}_n - \mathbf{\Psi}(\mathbf{z})\mathbf{N}] \neq 0;$$
$$\det\mathbf{W} = \det[\mathbf{I}_n - \mathbf{F}(\mathbf{g})] \neq 0. \tag{7}$$

(iii) If the vectors $\mathbf{q}$, $\mathbf{g}$ and matrices $\mathbf{D}$, $\mathbf{N}$ satisfy (5) and $\det \mathbf{V} \neq 0$, $\det \mathbf{W} \neq 0$, then the operator B_1 is correct and the unique solution of the problem $B_1 u = f$ is

$$u = B_1^{-1} f = \hat{A}^{-2} f + \mathbf{YF}(\hat{A}^{-1} f) + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-2} f) +$$
$$+ \left[\hat{A}^{-1}\mathbf{Y} + \mathbf{YF}(\mathbf{Y}) + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\mathbf{Y})\right]\mathbf{F}(f) +$$
$$+ \left[\hat{A}^{-1}\mathbf{z} + \mathbf{YF}(\mathbf{z}) + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\mathbf{z})\right]\mathbf{NV}^{-1}\mathbf{\Psi}(\hat{A}^{-1} f), \tag{8}$$

or

$$u = B_1^{-1} f = \hat{A}^{-1}\tilde{f} + \mathbf{YF}(\tilde{f}) + \mathbf{zNV}^{-1}\Psi(\hat{A}^{-1}\tilde{f}), \tag{9}$$

where

$$\tilde{f} = \hat{A}^{-1} f + \mathbf{YF}(f) + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1} f),$$
$$\mathbf{Y} = \left[\hat{A}^{-1}\mathbf{g} + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})\right]\mathbf{W}^{-1}. \tag{10}$$

Proof: (i) First we prove the second formula in (6). Denote by

$$\tilde{\mathfrak{D}} = \{u \in \mathfrak{D}(A^2) : \mathbf{\Phi}(u) = \mathbf{N\Psi}(u),\ \mathbf{\Phi}(Au) =$$
$$= [\mathbf{\Phi}(\mathbf{g}) - \mathbf{N\Psi}(\mathbf{g})]\mathbf{F}(Au) + \mathbf{N\Psi}(Au)\}.$$

Let $u \in \mathfrak{D}(B^2)$ and $\mathbf{g} \in \mathfrak{D}(A)^n$. Then by definition, $u \in \mathfrak{D}(B)$ and $Bu \in \mathfrak{D}(B)$, which since (3) implies $u \in \mathfrak{D}(A)$, $\mathbf{\Phi}(u) = \mathbf{N\Psi}(u)$ and $Bu \in \mathfrak{D}(A)$, $\mathbf{\Phi}(Bu) = \mathbf{N\Psi}(Bu)$. From $Bu = Au - \mathbf{gF}(Au) \in \mathfrak{D}(A)$ it follows that $u \in \mathfrak{D}(A^2)$. Further from the equation $\mathbf{\Phi}(Bu) = \mathbf{N\Psi}(Bu)$ is implied that $u \in \tilde{\mathfrak{D}}$.

Conversely, let $u \in \tilde{\mathfrak{D}}$, then $u \in \mathfrak{D}(A^2)$, $\mathbf{\Phi}(u) = \mathbf{N\Psi}(u)$ and $\mathbf{\Phi}(Au) - [\mathbf{\Phi}(\mathbf{g}) - \mathbf{N\Psi}(\mathbf{g})]\mathbf{F}(Au) = \mathbf{N\Psi}(Au)$. Then $u \in \mathfrak{D}(B)$, $Bu \in \mathfrak{D}(A)$ and $\mathbf{\Phi}(Au) - \mathbf{\Phi}(\mathbf{g})\mathbf{F}(Au) = \mathbf{N\Psi}(Au) + \mathbf{N\Psi}(\mathbf{g})\mathbf{F}(Au)$, which implies $\mathbf{\Phi}(Bu) = \mathbf{N\Psi}(Bu)$ or $Bu \in \mathfrak{D}(B)$. Hence $u \in \mathfrak{D}(B^2)$, and so (6) holds. Now we prove the first formula in (6). Let $u \in \mathfrak{D}(B^2)$, $y = Bu$, $\mathbf{g} \in \mathfrak{D}(A)^n$. Then

$$B^2 u = By = Ay - \mathbf{gF}(Ay) = ABu - \mathbf{gF}(ABu) =$$
$$= A[Au - \mathbf{gF}(Au)] - \mathbf{gF}(A[Au - \mathbf{gF}(Au)]) =$$

$$= A^2u - Ag\mathbf{F}(Au) - \mathbf{g}\mathbf{F}(A^2u) + \mathbf{g}\mathbf{F}(Ag)\mathbf{F}(Au).$$

Hence, $B_1 u = B^2 u$.

(ii) Let (5) holds and $\det\mathbf{V}$, $\det\mathbf{W} \neq 0$. By statement (i), $B_1 = B^2$ and so $\mathfrak{D}(B_1) = \mathfrak{D}(B^2)$. Since $\mathbf{\Phi}(\mathbf{z}) = \mathbf{I}_m$, the relations in (6) can be written as

$$\Phi(u - \mathbf{z}\mathbf{N}\Psi(u)) = 0,$$

$$\Phi(Au - [\mathbf{g} - \mathbf{z}\mathbf{N}\Psi(\mathbf{g})]\mathbf{F}(Au) - \mathbf{z}\mathbf{N}\Psi(Au)) = 0,$$

which taking into account (2) imply

$$u - \mathbf{z}\mathbf{N}\Psi(u) \in \mathfrak{D}(\hat{A}),$$

$$Au - [\mathbf{g} - \mathbf{z}\mathbf{N}\Psi(\mathbf{g})]\mathbf{F}(Au) - \mathbf{z}\mathbf{N}\Psi(Au) \in \mathfrak{D}(\hat{A}).$$

Then from (6), since $\mathbf{z} \in [\ker A]^m$, we obtain

$$\hat{A}\left(Au - \mathbf{g}\mathbf{F}(Au) + \mathbf{z}\mathbf{N}[\Psi(\mathbf{g})\mathbf{F}(Au) - \Psi(Au)]\right) + \mathbf{g}\left[\mathbf{F}(Ag)\mathbf{F}(Au) - \mathbf{F}(A^2u)\right] = f,$$

$$Au - \mathbf{g}\mathbf{F}(Au) + \mathbf{z}\mathbf{N}[\Psi(\mathbf{g})\mathbf{F}(Au) - \Psi(Au)] + \hat{A}^{-1}\mathbf{g}\left[\mathbf{F}(Ag)\mathbf{F}(Au) - \mathbf{F}(A^2u)\right] = \hat{A}^{-1}f,$$

$$\hat{A}(u - \mathbf{z}\mathbf{N}\Psi(u)) - \mathbf{g}\mathbf{F}(Au) + \mathbf{z}\mathbf{N}[\Psi(\mathbf{g})\mathbf{F}(Au) - \Psi(Au)] + \hat{A}^{-1}\mathbf{g}\left[\mathbf{F}(Ag)\mathbf{F}(Au) - \mathbf{F}(A^2u)\right] = \hat{A}^{-1}f,$$

and hence

$$u - \mathbf{z}\mathbf{N}\Psi(u) - \hat{A}^{-1}\mathbf{g}\mathbf{F}(Au) + \hat{A}^{-1}\mathbf{z}\mathbf{N}[\Psi(\mathbf{g})\mathbf{F}(Au) - \Psi(Au)] + \hat{A}^{-2}\mathbf{g}\left[\mathbf{F}(Ag)\mathbf{F}(Au) - \mathbf{F}(A^2u)\right] = \hat{A}^{-2}f,$$

and then

$$A^2u = [Ag - \mathbf{g}\mathbf{F}(Ag)]\mathbf{F}(Au) + \mathbf{g}\mathbf{F}(A^2u) + f,$$

$$Au = \mathbf{g}\mathbf{F}(Au) - \mathbf{z}\mathbf{N}[\Psi(\mathbf{g})\mathbf{F}(Au) - \Psi(Au)] - \hat{A}^{-1}\mathbf{g}\left[\mathbf{F}(Ag)\mathbf{F}(Au) - \mathbf{F}(A^2u)\right] + \hat{A}^{-1}f,$$

$$u = \mathbf{z}\mathbf{N}\Psi(u) + \hat{A}^{-1}\mathbf{g}\mathbf{F}(Au) - \hat{A}^{-1}\mathbf{z}\mathbf{N}[\Psi(\mathbf{g})\mathbf{F}(Au) - \Psi(Au)] - \hat{A}^{-2}\mathbf{g}\left[\mathbf{F}(Ag)\mathbf{F}(Au) - \mathbf{F}(A^2u)\right] + \hat{A}^{-2}f.$$

Acting by the vectors $\mathbf{F}$ and Ψ we get

$$\mathbf{F}(Au) = \mathbf{F}(\mathbf{g})\mathbf{F}(Au) - \mathbf{F}(\mathbf{z}) \times \mathbf{N}[\Psi(\mathbf{g})\mathbf{F}(Au) - \Psi(Au)] - \mathbf{F}(\hat{A}^{-1}\mathbf{g}) \times \left[\mathbf{F}(Ag)\mathbf{F}(Au) - \mathbf{F}(A^2u)\right] + \mathbf{F}(\hat{A}^{-1}f),$$

$$\Psi(Au) = \Psi(\mathbf{g})\mathbf{F}(Au) - \Psi(\mathbf{z}) \times \mathbf{N}[\Psi(\mathbf{g})\mathbf{F}(Au) - \Psi(Au)] - \Psi(\hat{A}^{-1}\mathbf{g}) \times \left[\mathbf{F}(Ag)\mathbf{F}(Au) - \mathbf{F}(A^2u)\right] + \Psi(\hat{A}^{-1}f),$$

$$\mathbf{V}\Psi(u) = \left[\Psi(\hat{A}^{-1}\mathbf{g}) - \Psi(\hat{A}^{-1}\mathbf{z})\mathbf{N}\Psi(\mathbf{g}) - \Psi(\hat{A}^{-2}\mathbf{g})\mathbf{F}(Ag)\right] \times \mathbf{F}(Au) + \Psi(\hat{A}^{-1}\mathbf{z})\mathbf{N}\Psi(Au) + \Psi(\hat{A}^{-2}\mathbf{g})\mathbf{F}(A^2u) + \Psi(\hat{A}^{-2}f),$$

$$\mathbf{F}(A^2u) = [\mathbf{F}(Ag) - \mathbf{F}(\mathbf{g})\mathbf{F}(Ag)]\mathbf{F}(Au) + \mathbf{F}(\mathbf{g})\mathbf{F}(A^2u) + \mathbf{F}(f),$$

or

$$\left[\mathbf{I}_n - \mathbf{F}(\mathbf{g}) + \mathbf{F}(\mathbf{z})\mathbf{N}\Psi(\mathbf{g}) + \mathbf{F}(\hat{A}^{-1}\mathbf{g})\mathbf{F}(Ag)\right] \times \mathbf{F}(Au) - \mathbf{F}(\hat{A}^{-1}\mathbf{g})\mathbf{F}(A^2u) - \mathbf{F}(\mathbf{z})\mathbf{N}\Psi(Au) = \mathbf{F}(\hat{A}^{-1}f),$$

$$\mathbf{V}\Psi(Au) - \left[\mathbf{V}\Psi(\mathbf{g}) - \Psi(\hat{A}^{-1}\mathbf{g})\mathbf{F}(Ag)\right]\mathbf{F}(Au) - \Psi(\hat{A}^{-1}\mathbf{g})\mathbf{F}(A^2u) = \Psi(\hat{A}^{-1}f),$$

$$\mathbf{V}\Psi(u) - \Psi(\hat{A}^{-1}\mathbf{z})\mathbf{N}\Psi(Au) - \left[\Psi(\hat{A}^{-1}\mathbf{g}) - \Psi(\hat{A}^{-1}\mathbf{z})\mathbf{N}\Psi(\mathbf{g}) - \Psi(\hat{A}^{-2}\mathbf{g})\mathbf{F}(Ag)\right] \times \mathbf{F}(Au) - \Psi(\hat{A}^{-2}\mathbf{g})\mathbf{F}(A^2u) = \Psi(\hat{A}^{-2}f),$$

$$[\mathbf{F}(Ag) - \mathbf{F}(\mathbf{g})\mathbf{F}(Ag)]\mathbf{F}(Au) + [\mathbf{F}(\mathbf{g}) - \mathbf{I}_n]\mathbf{F}(A^2u) = -\mathbf{F}(f).$$

Denoting

$$\mathbf{D}_1 = -\mathbf{V}\Psi(\mathbf{g}) + \Psi(\hat{A}^{-1}\mathbf{g})\mathbf{F}(Ag);$$

$$\mathbf{D}_2 = \mathbf{W} + \mathbf{F}(\mathbf{z})\mathbf{N}\Psi(\mathbf{g}) + \mathbf{F}(\hat{A}^{-1}\mathbf{g})\mathbf{F}(\hat{A}\mathbf{g});$$

$$\mathbf{D}_3 = -\Psi(\hat{A}^{-1}\mathbf{g}) + \Psi(\hat{A}^{-1}\mathbf{z})\mathbf{N}\Psi(\mathbf{g}) + \Psi(\hat{A}^{-2}\mathbf{g})\mathbf{F}(Ag),$$

we get

$$\begin{pmatrix} \mathbf{0}_n & -\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{D}_2 & -\mathbf{F}(\hat{A}^{-1}\mathbf{g}) \\ \mathbf{0}_n & \mathbf{V} & \mathbf{D}_1 & -\Psi(\hat{A}^{-1}\mathbf{g}) \\ \mathbf{V} & -\Psi(\hat{A}^{-1}\mathbf{z})\mathbf{N} & \mathbf{D}_3 & -\Psi(\hat{A}^{-2}\mathbf{g}) \\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{W}\mathbf{F}(Ag) & -\mathbf{W} \end{pmatrix} \times$$

$$\times\begin{pmatrix}\boldsymbol{\Psi}(u)\\ \boldsymbol{\Psi}(Au)\\ \mathbf{F}(Au)\\ \mathbf{F}\left(A^2u\right)\end{pmatrix}=\begin{pmatrix}\mathbf{F}\left(\hat{A}^{-1}f\right)\\ \boldsymbol{\Psi}\left(\hat{A}^{-1}f\right)\\ \boldsymbol{\Psi}\left(\hat{A}^{-2}f\right)\\ -\mathbf{F}(f)\end{pmatrix}. \quad (11)$$

Designating the matrix the left by $\mathbf{L}_2$, we have

$$\det\mathbf{L}_2=\det\begin{pmatrix}\mathbf{0}_n & -\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{D}_2 & -\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{0}_n & \mathbf{V} & \mathbf{D}_1 & -\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{V} & -\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{N} & \mathbf{D}_3 & -\boldsymbol{\Psi}\left(\hat{A}^{-2}\mathbf{g}\right)\\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{W}\mathbf{F}(A\mathbf{g}) & -\mathbf{W}\end{pmatrix}=$$

$$=\det\left[\begin{pmatrix}\mathbf{V}^{-1} & \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{V}^{-1} & \mathbf{0}_n & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{V}^{-1} & \mathbf{0}_n\\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n & \mathbf{W}^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{0}_n & -\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{D}_2 & -\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{0}_n & \mathbf{V} & \mathbf{D}_1 & -\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{V} & -\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{N} & \mathbf{D}_3 & -\boldsymbol{\Psi}\left(\hat{A}^{-2}\mathbf{g}\right)\\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{W}\mathbf{F}(A\mathbf{g}) & -\mathbf{W}\end{pmatrix}\right]|\mathbf{V}|^3|\mathbf{W}|=$$

$$=\det\begin{pmatrix}\mathbf{0}_n & -\mathbf{V}^{-1}\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{V}^{-1}\mathbf{D}_2 & -\mathbf{V}^{-1}\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{0}_n & \mathbf{I}_n & \mathbf{V}^{-1}\mathbf{D}_1 & -\mathbf{V}^{-1}\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{I}_n & -\mathbf{V}^{-1}\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{z}\right)\mathbf{N} & \mathbf{V}^{-1}\mathbf{D}_3 & -\mathbf{V}^{-1}\boldsymbol{\Psi}\left(\hat{A}^{-2}\mathbf{g}\right)\\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{F}(A\mathbf{g}) & -\mathbf{I}_n\end{pmatrix}|\mathbf{V}|^3|\mathbf{W}|=\pm\det\begin{pmatrix}-\mathbf{V}^{-1}\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{V}^{-1}\mathbf{D}_2 & -\mathbf{V}^{-1}\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{I}_n & \mathbf{V}^{-1}\mathbf{D}_1 & -\mathbf{V}^{-1}\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{0}_n & \mathbf{F}(A\mathbf{g}) & -\mathbf{I}_n\end{pmatrix}|\mathbf{V}|^3|\mathbf{W}|.$$

Multiplying from the right the third column of the determinant by the matrix **F(*A*g)** and adding to the second column, we obtain

$$\det\mathbf{L}_2=\pm\det\begin{pmatrix}-\mathbf{V}^{-1}\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{V}^{-1}\mathbf{D}_2-\mathbf{V}^{-1}\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{F}(A\mathbf{g}) & -\mathbf{V}^{-1}\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{I}_n & \mathbf{V}^{-1}\mathbf{D}_1-\mathbf{V}^{-1}\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\mathbf{F}(A\mathbf{g}) & -\mathbf{V}^{-1}\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{0}_n & \mathbf{0}_n & -\mathbf{I}_n\end{pmatrix}|\mathbf{V}|^3|\mathbf{W}|=$$

$$=\det\begin{pmatrix}-\mathbf{V}^{-1}\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{V}^{-1}\left[\mathbf{W}+\mathbf{F}(\mathbf{z})\mathbf{N}\boldsymbol{\Psi}(\mathbf{g})\right] & -\mathbf{V}^{-1}\mathbf{F}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{I}_n & -\boldsymbol{\Psi}(\mathbf{g}) & -\mathbf{V}^{-1}\boldsymbol{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\\ \mathbf{0}_n & \mathbf{0}_n & -\mathbf{I}_n\end{pmatrix}|\mathbf{V}|^3|\mathbf{W}|=$$

$$=\det\begin{pmatrix}-\mathbf{V}^{-1}\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{V}^{-1}\left[\mathbf{W}+\mathbf{F}(\mathbf{z})\mathbf{N}\boldsymbol{\Psi}(\mathbf{g})\right]\\ \mathbf{I}_n & -\boldsymbol{\Psi}(\mathbf{g})\end{pmatrix}|\mathbf{V}|^3|\mathbf{W}|.$$

Finally, multiplying from the right the first column of the determinant by the matrix **Ψ(g)** and adding to the second column we get

$$\det\mathbf{L}_2=\pm\det\begin{pmatrix}-\mathbf{V}^{-1}\mathbf{F}(\mathbf{z})\mathbf{N} & \mathbf{V}^{-1}\mathbf{W}\\ \mathbf{I}_n & \mathbf{0}_n\end{pmatrix}|\mathbf{V}|^3|\mathbf{W}|=$$

$$=\pm\left|\mathbf{V}^{-1}\right||\mathbf{W}||\mathbf{V}|^3|\mathbf{W}|=\pm|\mathbf{V}|^2|\mathbf{W}|^2.$$

So $\det\mathbf{L}_2 = \pm|\mathbf{V}|^2|\mathbf{W}|^2 \neq 0$. Let $u \in \ker B_1$. Then in (11) $f = 0$ and $\mathbf{L}_2\mathrm{col}(\mathbf{\Psi}(u), \mathbf{\Psi}(Au), \mathbf{F}(Au), \mathbf{F}(A^2u)) = \mathbf{0}$, which since $\det\mathbf{L}_2 \neq 0$, yields $\mathbf{\Psi}(u) = \mathbf{\Psi}(Au) = \mathbf{F}(Au) = \mathbf{F}(A^2u) = \mathbf{0}$. Substitution of these values into (6) imply $B_1u = B^2u = A^2u = 0$, $\mathbf{\Phi}(u) = \mathbf{\Phi}(Au) = \mathbf{0}$. Taking into account (2) we acquire $u \in \mathfrak{D}\left(\hat{A}^2\right)$ and $B_1u = \hat{A}^2u = 0$. By hypothesis $\hat{A}$ is correct and so $u = 0$. Thus $\ker B_1 = \{0\}$ and B_1 is injective.

Conversely, let $\det\mathbf{V} = 0$, then there exists a vector $\mathbf{c} = \mathrm{col}(c_1, ..., c_n) \neq \mathbf{0}$ such that $\mathbf{Vc} = 0$. Note that $u_0 = \mathbf{zNc} \neq 0$, otherwise, since the components of $\mathbf{z}$ are linearly independent, we have $\mathbf{Nc} = \mathbf{0}$ and from $\mathbf{Vc} = 0$ follows that $\mathbf{c} = 0$ wich contradicts the hypothesis $\mathbf{c} \neq 0$. Substituting u_0 into the first boundary condition (6), we get $\mathbf{\Phi}(u_0) - \mathbf{N\Psi}(u_0) = \mathbf{Nc} - \mathbf{N\Psi}(\mathbf{z})\mathbf{Nc} = \mathbf{N}[\mathbf{I}_n - \mathbf{\Psi}(\mathbf{z})\mathbf{N}]\mathbf{c} = \mathbf{NVc} = \mathbf{0}$. Substituting u_0 into the second boundary condition, we obtain $\mathbf{\Phi}(Au_0) - [\mathbf{\Phi}(\mathbf{g}) - \mathbf{N\Psi}(\mathbf{g})]\mathbf{F}(Au_0) - \mathbf{N\Psi}(Au_0) = 0$, since $\mathbf{z} \in \ker A$. So $u_0 \in \mathfrak{D}(B^2)$. It is evident that $u_0 \in \ker B^2$. Hence $u_0 \in \mathfrak{D}(B^2)$ and $u_0 \in \ker B^2$. So $\ker B^2 = \ker B_1 \neq \{0\}$ and $B^2 = B_1$ is not injective.

Let now $\det\mathbf{V} \neq 0$, but $\det\mathbf{W} = 0$. Then there exists a vector $\mathbf{c} = \mathrm{col}(c_1, ..., c_n) \neq \mathbf{0}$ such that $\mathbf{Wc} = 0$. Note that $\mathbf{gc} \neq 0$ because of $g_1, ..., g_n$ is a linearly independent set and that the element $u_0 = \left[\hat{A}^{-1}\mathbf{g} + \mathbf{zNV}^{-1}\mathbf{\Psi}\left(\hat{A}^{-1}\mathbf{g}\right)\right]\mathbf{c} \neq \mathbf{0}$, otherwise $\mathbf{c} = \mathbf{0}$. For u_0 we obtain

$$\mathbf{\Phi}(u_0) - \mathbf{N\Psi}(u_0) = \mathbf{NV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})\mathbf{c} - \mathbf{N\Psi}(\hat{A}^{-1}\mathbf{g})\mathbf{c} - \\ - \mathbf{N\Psi}(\mathbf{z})\mathbf{NV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})\mathbf{c} = \\ = \mathbf{N}[\mathbf{I}_n - \mathbf{\Psi}(\mathbf{z})\mathbf{N}]\mathbf{V}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})\mathbf{c} - \mathbf{N\Psi}(\hat{A}^{-1}\mathbf{g})\mathbf{c} = \\ = [\mathbf{N\Psi}(\hat{A}^{-1}\mathbf{g}) - \mathbf{N\Psi}(\hat{A}^{-1}\mathbf{g})]\mathbf{c} = \mathbf{0}.$$

So u_0 satisfies the first boundary condition (6). For u_0 we also obtain

$$\mathbf{\Phi}(Au_0) - [\mathbf{\Phi}(\mathbf{g}) - \mathbf{N\Psi}(\mathbf{g})]\mathbf{F}(Au_0) - \mathbf{N\Psi}(Au_0) = \\ = \mathbf{\Phi}(\mathbf{g})\mathbf{c} - [\mathbf{\Phi}(\mathbf{g}) - \mathbf{N\Psi}(\mathbf{g})]\mathbf{F}(\mathbf{g})\mathbf{c} - \mathbf{N\Psi}(\mathbf{g})\mathbf{c} = \\ = \mathbf{\Phi}(\mathbf{g})[\mathbf{I}_n - \mathbf{F}(\mathbf{g})]\mathbf{c} - \mathbf{N\Psi}(\mathbf{g})[\mathbf{I}_n - \mathbf{F}(\mathbf{g})]\mathbf{c} = \\ = \mathbf{\Phi}(\mathbf{g})\mathbf{Wc} - \mathbf{N\Psi}(\mathbf{g})\mathbf{Wc} = 0.$$

So $u_0 \in \mathfrak{D}(B^2)$. Moreover

$$B^2u_0 = A^2u_0 - [A\mathbf{g} - \mathbf{gF}(A\mathbf{g})]\mathbf{F}(Au_0) - \mathbf{gF}(A^2u_0) = \\ = A\mathbf{gc} - [A\mathbf{g} - \mathbf{gF}(A\mathbf{g})]\mathbf{F}(\mathbf{g})\mathbf{c} - \mathbf{gF}(A\mathbf{g})\mathbf{c} = \\ = A\mathbf{g}[\mathbf{I}_n - \mathbf{F}(\mathbf{g})]\mathbf{c} - \mathbf{gF}(A\mathbf{g})[\mathbf{I}_n - \mathbf{F}(\mathbf{g})]\mathbf{c} = \\ = A\mathbf{gWc} - \mathbf{gF}(A\mathbf{g})\mathbf{Wc} = 0.$$

Hence $u_0 \in \mathfrak{D}(B^2)$ and $u_0 \in \ker B^2$. So $\ker B^2 \neq \{0\}$ and B^2 is not injective. So we proved that B_1 is injective if and only if $\det\mathbf{V} \neq 0$, $\det\mathbf{W} \neq 0$. The statement (ii) holds.

(iii) Let the vectors $\mathbf{q}$, $\mathbf{g}$, $\mathbf{v}$, $\mathbf{w}$ and matrices $\mathbf{D}$, $\mathbf{N}$ satisfy (5) and $\det\mathbf{V} \neq 0$, $\det\mathbf{W} \neq 0$. Then, by the statement (ii), the operator $B_1 = B^2$ is injective and the problem $B_1u = f$ has a unique solution. We recall that by Theorem [1] the unique solution of the equation $Bu = f$ for all $f \in X$ is given by

$$u = B^{-1}f = \hat{A}^{-1}f + \\ + [\hat{A}^{-1}\mathbf{g} + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})]\mathbf{W}^{-1}\mathbf{F}(f) + \\ + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}f). \quad (12)$$

Let $B_1u = B^2u = f$, where $f \in X$. Denoting by $\mathbf{Y} = [\hat{A}^{-1}\mathbf{g} + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\mathbf{g})]\mathbf{W}^{-1}$ and $\tilde{f} = Bu$, we get $B\tilde{f} = f$. Then by means of (12) the solution of this equation is given by

$$\tilde{f} = B^{-1}f = \hat{A}^{-1}f + \mathbf{YF}(f) + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}f). \quad (13)$$

Applying again (12) we find the solution of the problem $Bu = \tilde{f}$ or $B_1u = f$

$$u = B_1^{-1}f = B^{-1}\tilde{f} = \hat{A}^{-1}\tilde{f} + \mathbf{YF}(\tilde{f}) + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\tilde{f}), \quad (14)$$

which is equation (9). Substituting the value of $\tilde{f}$ from (13) into (14), we get

$$u = B_1^{-1}f = B^{-2}f = B^{-1}\tilde{f} = \hat{A}^{-2}f + \hat{A}^{-1}\mathbf{YF}(f) + \\ + \hat{A}^{-1}\mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}f) + \mathbf{Y}[\mathbf{F}(\hat{A}^{-1}f) + \mathbf{F}(\mathbf{Y})\mathbf{F}(f) + \\ + \mathbf{F}(\mathbf{z})\mathbf{NV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}f)] + \mathbf{zNV}^{-1}[\mathbf{\Psi}(\hat{A}^{-2}f) + \\ + \mathbf{\Psi}(\hat{A}^{-1}\mathbf{Y})\mathbf{F}(f) + \mathbf{\Psi}(\hat{A}^{-1}\mathbf{z})\mathbf{NV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}f)] = \\ = \hat{A}^{-2}f + \mathbf{YF}(\hat{A}^{-1}f) + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-2}f) + \\ + [\hat{A}^{-1}\mathbf{Y} + \mathbf{YF}(\mathbf{Y}) + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\mathbf{Y})]\mathbf{F}(f) + \\ + [\hat{A}^{-1}\mathbf{z} + \mathbf{YF}(\mathbf{z}) + \mathbf{zNV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}\mathbf{z})]\mathbf{NV}^{-1}\mathbf{\Psi}(\hat{A}^{-1}f),$$

which is the solution (8). In the above solutions f is arbitrary, consequently, $R(B_1) = X$. Since the operators $\hat{A}^{-2}$, $\hat{A}^{-1}$ and the functionals $\mathbf{F}$ and $\mathbf{\Psi}$ are bounded, from (8) or (9) follows the boundedness of $B_1^{-1} = B^{-2}$, i. e. the operator B_1 is correct. The theorem is proved.

The next corollary follows from the above theorem in the case $\mathbf{q} = \mathbf{g} = \mathbf{0}$ and it is useful for solving a class of differential equations with multipoint and nonlocal boundary conditions.

Corollary. The differential operator $B_1: X \to X$ be defined by

$$B_1u = A^2u = f, \\ \mathfrak{D}(B_1) = \{u \in \mathfrak{D}(A^2): \mathbf{\Phi}(u) = \mathbf{N\Psi}(u), \\ \mathbf{\Phi}(Au) = \mathbf{N\Psi}(Au)\}. \quad (15)$$

Then the operator B_1 is correct if and only if

$$\det\mathbf{V} = \det[\mathbf{I}_n - \boldsymbol{\Psi}(\mathbf{z})\mathbf{N}] \neq 0 \qquad (16)$$

and the unique solution of (15) is given by

$$u = B_1^{-1}f = \hat{A}^{-2}f + \mathbf{zNV}^{-1}\boldsymbol{\Psi}(\hat{A}^{-2}f) + \\ + [\hat{A}^{-1}\mathbf{z} + \mathbf{zNV}^{-1}\boldsymbol{\Psi}(\hat{A}^{-1}\mathbf{z})]\mathbf{NV}^{-1}\boldsymbol{\Psi}(\hat{A}^{-1}f). \qquad (17)$$

Proof: For $\mathbf{g} = \mathbf{0}$ from (7) immediately follows $\det\mathbf{W} = \det\mathbf{I}_n = 1 \neq 0$. Then by Theorem, the operator B_1 is correct if and only if $\det\mathbf{V} \neq 0$. From (8) for $\mathbf{g} = \mathbf{q} = 0$ follows the solution of this problem.

Examples

In this section we consider some examples boundary value problems to explain the application of the decomposition-extension method and to demonstrate its efficiency.

First, we recall some known results. The problem

$$\hat{A}u = u^{(m)}(x) = f(x),$$
$$\mathfrak{D}(\hat{A}) = \{u \in \mathbb{C}^m[0, 1]:\ u(0) = \\ = u'(0) = \ldots = u^{m-1}(0) = 0\},$$

is correct and its exact solution is given by

$$u(x) = \hat{A}^{-1}f(x) = \frac{1}{(m-1)!}\int_0^x (x-t)^{m-1} f(t)\mathrm{d}t. \quad (18)$$

If the function $u(x) \in \mathbb{C}^m[a, b]$ and $x_0 \in [a, b]$, then the functionals $T_k(u) = u^{(k-1)}(x_0)$, $k = 1, \ldots, m$, and $T(u) = \sum_{k=1}^m a_k u^{(k-1)}(x_0)$ are linear and bounded on $\mathbb{C}^k[a, b]$ and $\mathbb{C}^m[a, b]$, respectively.

Example 1

Consider the differential boundary value problem $u''(x) = f(x)$, $x \in [0, 1]$:

$$u(0) = \nu u(1),\ u'(0) = \nu u'(1). \qquad (19)$$

By taking $X = \mathbb{C}[0, 1]$ and

$$Au = u'(x),\ \mathfrak{D}(A) = \{u \in \mathbb{C}^1[0, 1]\},$$
$$A^2u = u''(x),\ \mathfrak{D}(A^2) = \{u \in \mathbb{C}^2[0, 1]\};$$
$$\hat{A}u = Au,\ \ \mathfrak{D}(\hat{A}) = \{u \in \mathfrak{D}(A) : u(0) = 0\},$$
$$\hat{A}^2u = A^2u,\ \ \mathfrak{D}(\hat{A}^2) = \{u \in \mathfrak{D}(A^2) : u(0) = u'(0) = 0\},$$

we can put the problem (19) as in Corollary, equation (15), namely:

$$B_1u = A^2u = f,$$
$$\mathfrak{D}(B_1) = \{u \in \mathfrak{D}(A^2):\ u(0) = \nu u(1),\ u'(0) = \nu u'(1)\},$$

where $\boldsymbol{\Phi}(u) = (u(0))$, $\boldsymbol{\Psi}(u) = (u(1))$ and $\mathbf{N} = (\nu)$. Then $\boldsymbol{\Phi}(Au) = (u'(0))$ and $\boldsymbol{\Psi}(Au) = (u'(1))$. Let $\mathbf{z} = (1)$ and notice that $\boldsymbol{\Phi}(\mathbf{z}) = \mathbf{z}(0) = 1$. If $\det\mathbf{V} = \det(1 - \nu) \neq 0$ then the unique solution $u = B_1^{-1}f$ follows from Corollary where

$$\hat{A}^{-1}f = \int_0^x f(t)\mathrm{d}t,\ \ \hat{A}^{-2}f = \int_0^x (x-t)f(t)\mathrm{d}t,$$

by means of (18).

Example 2

Let the fourth-order differential boundary value problem with multipoint boundary conditions $u^{(4)}(x) = f(x)$, $x \in [0, 1]$:

$$u(0) = \nu_{11}u\left(\frac{1}{2}\right) + \nu_{12}u(1);$$
$$u'(0) = \nu_{21}u\left(\frac{1}{2}\right) + \nu_{22}u(1);$$
$$u''(0) = \nu_{11}u''\left(\frac{1}{2}\right) + \nu_{12}u''(1);$$
$$u'''(0) = \nu_{21}u''\left(\frac{1}{2}\right) + \nu_{22}u''(1). \qquad (20)$$

We recast the problem (20) into the form (15) where $X = \mathbb{C}[0, 1]$ and

$$Au = u''(x),\ \mathfrak{D}(A) = \{u \in \mathbb{C}^2[0, 1]\},$$
$$A^2u = u^{(4)}(x),\ \mathfrak{D}(A^2) = \{u \in \mathbb{C}^4[0, 1]\},$$
$$\hat{A}u = Au,\ \ \mathfrak{D}(\hat{A}) = \{u \in \mathfrak{D}(A) : u(0) = u'(0) = 0\},$$
$$\hat{A}^2u = A^2u,$$
$$\mathfrak{D}(\hat{A}^2) = \{u \in \mathfrak{D}(A^2) : u(0) = u'(0) = u''(0) = u'''(0) = 0\};$$

$$\boldsymbol{\Phi}(u) = \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix};\ \ \boldsymbol{\Psi}(u) = \begin{pmatrix} u\left(\frac{1}{2}\right) \\ u(1) \end{pmatrix};$$

$$\boldsymbol{\Phi}(Au) = \begin{pmatrix} u''(0) \\ u'''(0) \end{pmatrix};\ \ \boldsymbol{\Psi}(Au) = \begin{pmatrix} u''\left(\frac{1}{2}\right) \\ u''(1) \end{pmatrix};$$

$$\boldsymbol{\Phi}(u) = \mathbf{N}\boldsymbol{\Psi}(u) = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix}\boldsymbol{\Psi}(u);\ \ \boldsymbol{\Phi}(Au) = \mathbf{N}\boldsymbol{\Psi}(Au).$$

Let $\mathbf{z} = (1, x)$ and notice that $\boldsymbol{\Phi}(\mathbf{z}) = \mathbf{z}(0) = \mathbf{I}_2$. If

$$\det\mathbf{V} = \det\left(\mathbf{I}_2 - \begin{pmatrix} 1 & 1/2 \\ 1 & 1 \end{pmatrix}\mathbf{N}\right) \neq 0,$$

then the unique solution $u = B_1^{-1}f$ follows from Corollary by making use of (18).

Example 3

Contemplate the fourth-order Fredholm integro-differential boundary value problem with general integral boundary conditions

$$u^{(4)}(x)+\left(x^2-1\right)\int_0^1 xu''(x)\mathrm{d}x-\left(x^2+1\right)\times$$
$$\times\int_0^1 xu^{(4)}(x)\mathrm{d}x=x-2,$$
$$x\in[0,1],$$
$$u(0)=\frac{15}{22}\int_0^1 u(x)\mathrm{d}x;$$
$$u'(0)=0;$$
$$u''(0)=\frac{1}{11}\int_0^1 xu''(x)\mathrm{d}x+\frac{15}{22}\int_0^1 u''(x)\mathrm{d}x;$$
$$u'''(0)=0. \quad (21)$$

We formulate the problem (21) as in (4). We take $X=\mathbb{C}[0,1]$ and

$$Au=u''(x),\ \mathfrak{D}(A)=\{u\in\mathbb{C}^2[0,1]\},$$
$$A^2u=u^{(4)}(x),\ \mathfrak{D}(A^2)=\{u\in\mathbb{C}^4[0,1]\},$$
$$\hat{A}u=Au,\quad \mathfrak{D}(\hat{A})=\{u\in\mathfrak{D}(A):u(0)=u'(0)=0\},$$
$$\hat{A}^2u=A^2u,$$
$$\mathfrak{D}(\hat{A}^2)=\{u\in\mathfrak{D}(A^2):u(0)=u'(0)=u''(0)=u'''(0)=0\};$$
$$f(x)=x-2;$$
$$\mathbf{q}=(1-x^2);\ \mathbf{g}=(x^2+1);$$
$$\mathbf{F}(Au)=\left(\int_0^1 xu''(x)\mathrm{d}x\right);\ \mathbf{F}\left(A^2u\right)=\left(\int_0^1 xu^{(4)}(x)\mathrm{d}x\right);$$
$$\mathbf{\Phi}(u)=\begin{pmatrix}u(0)\\ u'(0)\end{pmatrix};\ \mathbf{\Psi}(u)=\left(\int_0^1 u(x)\mathrm{d}x\right);$$
$$\mathbf{\Phi}(Au)=\begin{pmatrix}u''(0)\\ u'''(0)\end{pmatrix};\ \mathbf{\Psi}(Au)=\left(\int_0^1 u''(x)\mathrm{d}x\right);$$

$$\mathbf{\Phi}(u)=\mathbf{N\Psi}(u)=\begin{pmatrix}15/22\\ 0\end{pmatrix}\mathbf{\Psi}(u);$$
$$\mathbf{\Phi}(Au)=\mathbf{DF}(Au)+\mathbf{N\Psi}(Au)=$$
$$=\begin{pmatrix}1/11\\ 0\end{pmatrix}\mathbf{F}(Au)+\mathbf{N\Psi}(Au).$$

Observe that $\mathbf{g}=(x^2+1)\in\mathfrak{D}(A)$ and $\mathbf{q}=A\mathbf{g}-\mathbf{gF}(A\mathbf{g})$,

$$\mathfrak{D}=\mathbf{\Phi}(\mathbf{g})-\mathbf{N\Psi}(g). \quad (22)$$

Let $\mathbf{z}=(1,x)$. Then $\mathbf{\Phi(z)}=\mathbf{I}_2$, $\mathbf{\Psi}(\mathbf{z})=\left(1,\ \frac{1}{2}\right)$ and

$$\det\mathbf{V}=\det\left(\mathbf{I}_1-\mathbf{\Psi}(\mathbf{z})\mathbf{N}\right)=\frac{7}{22}\neq 0;$$
$$\det\mathbf{W}=\det\left(\mathbf{I}_1-\mathbf{F}(\mathbf{g})\right)=\frac{1}{4}\neq 0. \quad (23)$$

Bacause of (22) and (23), Theorem applies. Hence the operator B_1 is correct which means that the problem (21) admits a unique solution. By substituting into (8) or (9) and making use of (18), we get

$$u(x)=-\frac{1}{317\,520}\left(2352x^6-2646x^5+\right.$$
$$\left.+212\,387x^4+1169\,427x^2+926\,103\right).$$

Conclusion

By means of decomposition and the extension method, we provided a ready to use formula for constructing the solution in closed form of boundary value problems involving the composite square of an m^{th} order integro-differential operator of Fredholm type and nonlocal boundary conditions such as appropriate multipoint and integral conditions. The method is also applicable to boundary value problems for the composite squared m^{th} order linear ordinary differential operators.

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Метод нахождения точных решений для интегро-дифференциальных уравнений Фредгольма с мультиточечными и интегральными краевыми условиями. Часть 2. Метод разложения-расширения квадратичных операторов

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Введение: в первой части статьи представлен прямой метод исследования проблемы разрешимости и единственности и получения в замкнутой форме решения краевых задач, включающих линейный обыкновенный интегро-дифференциальный оператор Фредгольма или дифференциальный оператор m-го порядка, а также многоточечные и интегральные граничные условия. Здесь мы сосредоточимся на специальном классе краевых задач, включающих квадрат интегро-дифференциального оператора и соответствующих нелокальных граничных условий. **Цель:** исследование построения единственного решения краевых задач 2-го порядка в частном случае оператора, который может быть представлен в виде композиции квадратов операторов более низких порядков, а также разработка алгоритма построения точного решения в этом частном случае. **Результаты:** с помощью декомпозиции и метода расширения, описанного в первой части, нами разработан алгоритм для получения точного решения краевых задач для квадро-интегро-дифференциальных операторов или дифференциальных операторов с многоточечными и интегральными граничными условиями. Этот метод прост в использовании и может быть легко имплементирован в большинство современных систем компьютерной алгебры.

Ключевые слова — дифференциальные и фредгольмовы интегро-дифференциальные уравнения, многоточечные и нелокальные интегральные граничные условия, разложение операторов, корректные операторы, точные решения.

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Страница	Столбец	Строка	Напечатано	Следует читать
15	правый	8 снизу	of (5) is	of (5) for $\alpha_0 = 1, \alpha_1 = \dots = \alpha_m = 0, a = 0$ is
15	правый	6 снизу	$f(x) \in \mathbb{C}[a, b]$.	$f(x) \in \mathbb{C}[0, b]$.
16	левый	9 снизу	X^{m-1} and respectively	$[X^{m-1}]^*$ and X^* respectively